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CONNECTING WAZEWSKI'S CONDITIONS WITH –M-MATRICES: APPLICATION TO CONSTRAINED STABILIZATION.

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ABSTRACT: This paper connects Wazewski's conditions with –M-matrices which are both involved when dealing with Lyapunov stability using overvaluing comparison techniques. Then it provides methods for stabilization of a class of time-continuous systems under constraints on both control and state vector. An illustrative example is developed.

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1. INTRODUCTION

Basic results concerning differential inequalities were provided by Wazewski [25], who gave hypotheses ensuring that the solution of a system described by:

$\frac{dx}{dt} = f(t, x)$, with initial condition x_0 at time t_0 and function f verifying the inequality $f(t, x) \le g(t, x)$, is overvalued by the solution of the so-called "overvaluing system":

$\frac{dz}{dt} = g(t, z)$, with initial condition $z_0 \ge x_0$ at time t_0, or in other words, conditions on function $g(t, x)$ that guaranty $x(t) \le z(t)$ if $x(t_0) \le z(t_0)$.

These "Wazewski's conditions" appear to be fundamental ones when using comparison techniques [9].

On the other hand, the special class of matrices with non-positive off-diagonal elements and positive principal minors, that are called M-matrices [7], and their opposite that are called –M-matrices, presents many applications to convergence studies, in particular when dealing with vector-Lyapunov functions involving Vector Norms (V.N) techniques [6, 8, 11], and for constrained control [5, 20, 21, 22].

By this way, the stability theory literature obviously suggests a tight connection between Wazewski's conditions and –M-matrices. However, this always appears in an implicit maner and the present work aims at clarifying the existing implications.

A second part of the paper applies this to constrained control and positively invariant sets: on the one hand, for linear systems with linear state feedback control, the problem of

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the stabilization under constraints on control and/or state is quasi-solved (see for example [3, 5, 23, 24]). On the other hand, it remains an open question for general systems described by:

$$\frac{dx}{dt} = f(t, x, d, c),\ c = K\,x, \qquad (LSF)\ (1)$$

where:
$t \in \mathbb{R}$ is the **time variable**,
$x \in \mathbb{R}^n$ is the **state vector**,
$d \in S_D$, is the **disturbance vector**, that represents disturbances or unknown modelling parameters,
$S_D \subset \mathbb{R}^r$, the **set of admitted disturbances**,
$\underline{x}(t\,;\,t_0, x_0\,;\,d\,;\,c)$ is the **system motion** (in short $\underline{x}(t)$),
$c \in \mathbb{R}^m$ is the **control vector**, $K \in \mathbb{R}^{m \times n}$.

For system (LSF) (1), Radhy [20, 21, 22] contributes to this problem using V.N: a linear state feedback control is computed using an optimization procedure and a linear function g.

In this paper, we use non linear overvaluing function g, which appears to be less conservative.

2. NOTATIONS

In the following, we also consider the system:

$$\frac{dx}{dt} = f(t, x, d, u), \qquad (S)\ (2)$$

where: $u \in \mathbb{R}^m$ is the control vector of a more general form than in (LSF) (1), and we assume that the solutions $\underline{x}(t)$ of either (1) or (2) exist, are unique and continuous w.r.t. time t, $t \in [t_0, +\infty[$.

- for $a = [\alpha_1, \ldots, \alpha_l]^T \in \mathbb{R}^l, |a| = [|\alpha_1|, \ldots, |\alpha_l|]^T$.
- $a < b$, (a and $b \in \mathbb{R}^l$), elementwise inequality.
- Let $\mathbb{R} = \mathbb{R}^{n_1} \oplus \mathbb{R}^{n_2} \oplus \ldots \oplus \mathbb{R}^{n_k}$, $\sum_{i=1}^{k} n_k = n$, then P_i denotes the projection operator from $\mathbb{R}^n$ onto $\mathbb{R}^{n_i}$: $x_i = P_i\,x$, $x_i = [x_{i1}, \ldots, x_{in_i}]^T$, $x_i \in \mathbb{R}^{n_i}$.
- $p(x)$ a regular Vector Norm (V.N) of size k with components $p_i(x_i)$ scalar norms of the x_i, $p(x) = [p_1(x_1), \ldots, p_k(x_k)]^T$.
- $D_t^+ p_i(x) = D_t^+ p_i(x_i) = \lim_{\theta \to 0^+} \frac{p_i[x_i(t+\theta)] - p_i[x_i(t)]}{\theta}$, the right-hand time derivative (Dini derivative).

• $D^+_{x_i} p_i(x) = D^+_{x_i} p_i(x_i)$, the i^{th} vector element of the right-hand gradient.

$D^+_{x_i} p_i(x_i) = \left[D^+_{x_{i1}} p_i(x_i), \ldots, D^+_{x_{in_i}} p_i(x_i) \right]^T$,

$D^+_{x_{ij}} p_i(x_i) = \underset{\substack{\theta \to 0^+ \\ |\Delta x_i| \to 0}}{\text{Lim}} \frac{p_i[x_i + I_{i,j+1} \Delta x_i] - p_i[x_i + I_{i,j} \Delta x_i]}{\Delta x_i}$,

$\Delta x_i = x_i(t) - x_i(t + \theta)$,

$I_{i,j} = \text{Diag}\ \{(1-\delta_{1j}), \ldots, (1-\delta_{ij}), 0, \ldots, 0\}$, $I_{i,j} \in \mathbb{R}^{n_i \times n_i}$, $\delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$.

• $\mathcal{C}^\alpha(\mathcal{S}, \mathbb{R}^k)$, the set of α-times continuously differentiable functions from $\mathcal{S}$ into $\mathbb{R}^k$.

• $\mathcal{X}_w = \{x \in \mathbb{R}^n : p(x) < w\}$, with $w > 0$, $w \in \mathbb{R}^k$, a bounded connected open neighbourhood of $x = 0$.

• $\overline{(.)}$, $\overset{\circ}{(.)}$, $\partial\,(.)$ respectively denote the closure, the interior and the boundary of the set (.).

• ρ, the Euclidean distance.

3. OVERVALUING COMPARISON SYSTEMS

3.1. Definitions and Lemma

Let $\mathcal{S}$ be a subset of $\mathbb{R}^k$. Let $g \in \mathcal{C}^1(\mathcal{S}, \mathbb{R}^k)$ and $V \in C^0(\mathbb{R}^n, \mathcal{S})$ be a function of the variable x which may be a motion of (2) or (1). Let us suppose that the right hand time derivative of $V(x)$ satisfies the following inequality system:

$$D^+_t V(x) \leq g(V(x)),\ V(x) \in \mathcal{S}. \tag{3}$$

Then we associate with (3), the following **overvaluing system (O.S.)** of (3) described by:

$$\frac{dz}{dt} = g(z),\ z \in \mathcal{S}. \tag{4}$$

Definition 1 (see [13, 18])

i) $g : z \to g(z)$, is **locally quasi-monotone non-decreasing w.r.t.** $\mathcal{S}$ if and only if:

$\forall\, i \in \{1, \ldots, k\}$ and $(x, y) \in \mathcal{S} \times \mathcal{S}$, such that: $x_i = y_i$ and $x_j \leq y_j$ $(j \neq i)$, the inequality $g_i(x) \leq g_i(y)$ holds. (5)

ii) if in i) $\mathcal{S} = \mathbb{R}^k$, then $g(z)$ is quasi-monotone non-decreasing.

Remarks 1:

1) This definition also appears in a more general form where g includes the time variable and other variables (say $g(t, x, y)$) and satisfys a similar form to (5) with respect to one variable only (say x) (see [1, 2, 15]).

2) The second point ii) of this definition is a special case of the more general definition of "mixed quasimonotonicity" where some components of g are quasi-monotone non-decreasing and the other are quasi-monotone non-increasing (see [13]).

3) Functions that satisfy the Wazewski's conditions received different names with sometimes slight differences for the continuity:

- "increasing with regard to the diagonal elements" ([12]),
- "uniformly non singular monotone" (see [9] p.117),

if in addition g is requested to be continuous:

- "property K or H" which are equivalent under certain hypotheses concerning the set relative to the property (see [25]).

Lemma 1

Let us suppose that:

1) for a given function $V \in \mathcal{C}^0(\mathbb{R}^n, \mathcal{S})$ there exists a function locally quasi-monotone non-decreasing w.r.t. $\mathcal{S}$: $g \in \mathcal{C}^1(\mathcal{S}, \mathbb{R}^k)$ satisfying (3),

2) $\mathcal{U} \subset \mathcal{S}$, is a positively invariant set w.r.t. (4).

Then:

1) $\forall\, z_0 \in \mathcal{U}$, $\underline{z}(t\,;\,t_0, z_0)$ the solution of (4) passing through z_0 at t_0 is unique, continuous w.r.t. time t and defined $\forall\, t \in [t_0, +\infty[$,

2) $\forall\, d \in S_D, \forall\, z_0 \in U : 0 \le V(x_0) \le z_0$, the inequality:

$0 \le V[\underline{x}(t)] \le \underline{z}(t\,;\,t_0, z_0)$, holds $\forall\, t \in [t_0, +\infty[$,

3) $\forall\, d \in S_D$, $\forall\, x_0 : V(x_0) \in \mathcal{U}$, $V[\underline{x}(t)] \in \mathcal{U}$ for $t \in [t_0, +\infty[$. ■

Proof This is a classical result [25], except that it applies to a restricted set $\mathcal{U}$ (see [18]).

Remark 2:

If $g = O\, V(x)+o$, where O has non-negative off diagonal elements, then g is quasi-monotone non-decreasing and thus the conclusions of lemma 1 hold.

3.2. Use of Vector Norms

Consider a defined control law $c(t, x)$ and let us rewrite system $(\mathcal{S})$ (2) as:

$$\frac{dx}{dt} = A(t, x, d)\, x + b(t, x, d). \tag{6}$$

Let us consider the k-sized regular Vector Norm (V.N.) $p(x)$ and let us define, for any i and j from the set $\{1, ..., k\}$:

$$x \in \mathbb{R}^n \text{ and } y \in \mathbb{R}^n,\ x_i = P_i\, x,\ y_i = P_i\, y. \tag{7}$$

$$b_i(t, x, d) = P_i\, b(t, x, d),\ t \in \mathbb{R},\ x \in \mathbb{R}^n,\ d \in S_D,$$

$$o_i(x) = \sup_{\substack{t \in \mathbb{R} \\ d \in S_D}} |[D^+_{x_i} p_i(x_i)]^T b_i(t, x, d)|,$$

$$o(x) = [o_i(x)]^T. \tag{8}$$

$$A_{ij}(t, x, d) = P_i A(t, x, d) P_j,$$

$$m_{ij}(t, x, y, d) = \frac{[D^+_{y_i} p_i(y_i)]^T A_{ij}(t, x, d) y_j}{p_j(y_j)},$$

$$o_{ij}(x) = \sup_{\substack{t \in \mathbb{R} \\ y \in \mathbb{R}^n \\ d \in S_D}} m_{ij}(t, x, y, d),$$

$$O(x) = [o_{ij}(x)]. \tag{9}$$

We assume that the above-defined terms are meaningful. For classical Holder-type norms, the computation of $(O(x), o(x))$ is easy to implement (see [4, 6, 8, 18]). For example, system (6) with V.N. $p(x) = |x| = [|x_1| \ldots |x_n|]^T$, yields:

$$O(x) = \sup_{\substack{t \in \mathbb{R} \\ d \in S_D}} \begin{bmatrix} \cdot & |\cdot| & |\cdot| \\ |\cdot| & a_{ii} & |\cdot| \\ |\cdot| & |\cdot| & \cdot \end{bmatrix} \text{(elementwise)}, \tag{10}$$

$$o(x) = \sup_{\substack{t \in \mathbb{R} \\ d \in S_D}} \begin{bmatrix} |\cdot| & |b_i| & |\cdot| \end{bmatrix}^T \text{(elementwise)}. \tag{11}$$

Lemma 2

The following inequality holds for system (6):

$$D^+_t p(x) \leq O(x)\, p(x) + o(x), \tag{12}$$

if $O(x)$ and $o(x)$ are respectively defined by (9), (8). ∎

Proof omitted for sake of brevity (see [18] p.147)

Remark 3:

There generaly exists a V.N. p (in particular $p(x) = |x|$) allowing to rewrite the right part of (12) as a function of $p(x)$: $g(p(x))$, or such that:

$$O(x)\, p(x) + o(x) \leq g(p(x)),\ p(x) \in S. \tag{13}$$

Thus Lemma 1 can be applied.

3.3. Connecting Wazewski's conditions with opposite of M-matrices

Using Remark 3, we can compute O.S. via V.N.

Let us suppose that:

1) the V.N. leads to inequality (3) with $g \in C^1(\mathcal{S}, \mathrm{IR}^k)$,

2) system (4) admits an equilibrium point $z_e \in \mathcal{S}$, and $\mathcal{S}$ is a connected neighbourhood of z_e.

Let $A(z) = \left[\dfrac{\partial g_i(z)}{\partial z_j}\right]$ be the Jacobian Matrix of g at z, and $A = A(z_e)$.

If we state the following assertions:

P1: g is locally quasi-monotone non-decreasing w.r.t. a neighbourhood ($\mathcal{N}$) of z_e,

P2: $A : a_{ij} \geq 0,\ j \neq i$,

P2': $A : a_{ij} > 0,\ j \neq i$,

P2'': $\dfrac{\partial g_i(z)}{\partial z_j} > 0$ on $\mathcal{N}$,

P3: z_e is hyperbolic (with no center eigenspace) and asymptotically stable,

P4: A is a –M-matrix (i.e $a_{ij} \geq 0,\ \forall\, j \neq i$ and every eigenvalue of A has a strictly negative real part),

then, the following lemma connects Wazewski's conditions with –M-matrices:

Lemma 3

P1 $\Rightarrow$ P2,

P2' $\Rightarrow$ P1,

P2'' $\Rightarrow$ P1,

[P1 and P3] $\Rightarrow$ P4. ■

Proof

P1 $\Rightarrow$ P2:

First, without loss of generality put $z_e = 0$ (because a change of coordinates shows that the obtained right part of (4) satisfies P1 and the general asumptions, with $z_e = 0$). Now, if the implication is false then $\exists\,(i, j)$ such that $a_{ij} < 0$. So, let $\varepsilon > 0$ (arbitrarly small) and $e = \varepsilon\,[0 \ldots 1 \ldots 0]^T$ (1 at the j^{th} place), using condition P1 we show that $\forall\, j \neq i$, $g_i(e) \geq 0$. $g \in C^1(\mathcal{S}, \mathrm{IR}^k)$, so using a Taylor expansion: $\exists\, \delta > 0$ such that $\forall\, \varepsilon$ such that $\delta > \varepsilon > 0$, $\mathrm{sign}(g_i(e)) = \mathrm{sign}(a_{ij}) = +1$ (contradiction).

P2' $\Rightarrow$ P1:

Let $i \in \{1, \ldots, k\}$, x and y two distinct points of $\mathcal{S}$, such that: $x_i = y_i$ and $x_j \leq y_j$, with $j \neq i$. Consider the following set: $S_{ij} = \{j \neq i : x_j < y_j\}$, this set is not empty, thus:

$\sum_{j \neq i} a_{ij}(y_j - x_j) = \sum_{j \in S_{ij}} a_{ij}(y_j - x_j) > 0$. So, there exists a neighbourhood $\mathcal{N} \subset \mathcal{S}$ of z_e, such that: $\text{sign}(g_i(y) - g_i(x)) = \text{sign}(\sum a_{ij}(y_j - x_j)) = +1$. Thus g is locally ($\mathcal{N}$) quasi-monotone non-decreasing.

P2'' $\Rightarrow$ P1:

Let $i \in \{1, ..., k\}$, x and y two distinct points of $\mathcal{S}$, such that: $x_i = y_i$ and $x_j \leq y_j$, with $j \neq i$. Consider the following function: ϕ: $[0, 1] \rightarrow \mathcal{S} \rightarrow \mathbb{R}$

$$\lambda \rightarrow z(\lambda) = x + \lambda(y - x) \rightarrow \phi(\lambda) = g_i(z(\lambda)).$$

Function ϕ is also C^1, thus on one hand $I = \int_0^1 \phi'(\lambda)\, d\lambda = \phi(1) - \phi(0) = g_i(y) - g_i(x)$

and on the other $I = \int_0^1 \phi'(\lambda)\, d\lambda = \int_0^1 \sum_{j \neq i} \frac{\partial g_i(z)}{\partial z_j}\Big|_{z(\lambda)} (y_j - x_j)\, d\lambda$, which is positive.

[P1 and P3] $\Rightarrow$ P4: it comes directly from P1 $\Rightarrow$ P2.

4. APPLICATION TO CONSTRAINED CONTROL

4.1. Problem formulation

Consider system (LSF) (1) that has to reach a final target set $\mathcal{A}$, subjected to constraints on both state and control related to two V.N. $p_1(x)$ and $p_2(c)$ (see [20, 23]):

1) x is constrained to belong to S_{SC}:

$$S_{SC} = \{x \in \mathbb{R}^n : p_1(x) \leq \text{sc}\}, \tag{14}$$

2) c is constrained to belong to S_{CC}:

$$S_{CC} = \{c \in \mathbb{R}^o : p_2(c) \leq \text{cc}\}. \tag{15}$$

In practice, most of the time, the two V.N. $p_1(x)$ and $p_2(c)$ are respectively: $p_1(x) = |x|$ and $p_2(c) = |c|$.

Here, the particular choice of the control (linear state-feedback) implies the reformulation of (15) as a constraint on the **gain matrix** K:

$$S_{KC} = \{K \in \mathbb{R}^{o \times n} : p_2(K\,x) \leq \text{cc with } x \in S_{SC}\},$$

or, as a constraint on the state vector x:

$$S_{SC}(K) = \{x \in \mathbb{R}^n : p_2(K\,x) \leq \text{cc}\}. \tag{16}$$

Thus the problem can be formulated as follow:

Problem: $\mathcal{P}((LSF)\ (1), \mathcal{A}, S_{SC}, S_{SC}(K))$
For the given sets $\mathcal{A}$, S_{SC} and $S_{SC}(K)$, find a gain matrix K such that: $\forall\ d \in S_D, \forall\ t_0 \in \mathbb{R}, \forall\ x_0 \in (S_{SC} \cap S_{SC}(K))$, the two following properties hold:
P1) $\underline{x}(t\ ; t_0, x_0\ ; d\ ; K) \in (S_{SC} \cap S_{SC}(K))$, for every $t \in [t_0, +\infty[$,
P2) $\lim_{t \to +\infty} \rho(\mathcal{A}\ ; \underline{x}(t\ ; t_0, x_0\ ; d\ ; K)) = 0$.

When $\mathcal{P}$ can't be solved exactly, it is reasonable to change the sets $\mathcal{A}$ and $(S_{SC} \cap S_{SC}(K))$.

4.2. Theorems

As previously seen the two constraints can be reduced to only one $(S_{SC} \cap S_{SC}(K))$. Let us suppose that: $(S_{SC} \cap S_{SC}(K)) = S_{SC}$, thus we do not use the V.N $p_2(x)$, and choose $p(x) = p_1(x)$ for applying results of part 3, with $p(x) = V(x) = p_1(x)$. This leads to:

$$D_t^+ p(x) \leq g(p(x)),$$

where g is defined by (13). In this part, we use the notation "g^K" to recall that the obtained function g depends on the gain matrix K.

At this level, in order to solve our problem, it is of importance to obtain results giving sufficient conditions (on K) which ensure the positive invariance of $(S_{SC} \cap S_{SC}(K))$ with respect to $\{(LSF)\ (1), S_D\}$ in the sense of the following definition:

Definition 2

A connected set C is **positively invariant** w.r.t. $\{(LSF)\ (1), S_D\}$ if and only if:

$$\forall\ d \in S_D, \forall\ x_0 \in C, \underline{x}(t) \in C \text{ for every } t \in [t_0, +\infty[. \tag{17}$$

Lemma 4 (Invariance lemma)

Let us suppose that:

1) the V.N. leads to inequality (3) with $g^K \in C^1(S, \mathbb{R}^k)$ locally quasi-monotone non-decreasing w.r.t. S,

2) system (4) admits a positive hyperbolic equilibrium point z_e, which has a domain of asymptotic stablility (see [10, 18]), $\mathcal{D}_{as}(z_e) \neq \emptyset$ with $\{z \in \mathbb{R}_+^k : z \leq z_e\} \subsetneq \mathcal{D}_{as}(z_e) \subset S$,

3) $X_w = \{x \in \mathbb{R}^n : p(x) < w, w > z_e : g^K(w) < 0\}$.

Then:

1) Such $\mathcal{X}_w$ exist (i.e $\exists\, w > z_e : g^K(w) < 0$),

2) Every $\overline{\mathcal{X}}_w$ with $(\{z \in \mathbb{R}_+^k : z \leq w\} \subset \mathcal{D}_{as}(z_e))$ is positively invariant w.r.t. $\{(LSF)\ (1),\ S_D\}$,

3) Every finite intersection or union of such $\overline{\mathcal{X}}_w$ with $(\{z \in \mathbb{R}_+^k : z \leq w\} \subset \mathcal{D}_{as}(z_e))$ is positively invariant w.r.t. $\{(LSF)\ (1),\ S_D\}$. ■

Proof

1) First, without loss of generality put $z_e = 0$. Let A be the Jacobian matrix of g^K at $z_e = 0$, then lemma 3 shows that A is the opposite of an M-matrix. Thus $\exists\, u > 0$ such that: $A\, u < 0$ [7]. Let $\varepsilon > 0$ (arbitrarly small) and $c = \varepsilon\, u$, then using a Taylor expansion, $g^K \in C^1(S, \mathbb{R}^k)$, $\exists\, \delta > 0$ such that:

$\forall\, \delta > \varepsilon > 0$, $\text{sign}(g_i^K(w)) = -1$ ($\forall\, i$). This proves 1).

2) Let $x \in \partial \overline{\mathcal{X}}_w$, then $\exists\, i \in \{1, 2, \ldots, k\}$ such that:

$$p(x) = \begin{bmatrix} p_1(x) \leq \chi_1 \\ \cdots \\ p_i(x) = \chi_i \\ \cdots \\ p_k(x) \leq \chi_k \end{bmatrix}, \text{ with } w = \begin{bmatrix} \chi_1 \\ \cdots \\ \chi_i \\ \cdots \\ \chi_k \end{bmatrix}.$$

As $x \in \overline{\mathcal{X}}_w$ and $(\{z \in \mathbb{R}_+^k : z \leq w\} \subset \mathcal{D}_{as}(z_e) \subset S)$:

$D_t^+ p_i(x) \leq g_i^K(x) \leq g_i^K(w) < 0$ (using condition 1) of lemma 4). This prove 2).

3) It is obvious.

Remark 4:

This invariance property of $\overline{\mathcal{X}}_w$ can be involved in order to obtain new kinds of estimates of $\mathcal{D}_{as}(z_e)$.

Theorem 1

Let us suppose that there exists $\hat{K}$ such that:

1) the V.N. leads to (3) with $g^{\hat{K}} \in C^1(S, \mathbb{R}^k)$ locally quasi-monotone non-decreasing w.r.t. S,

2) (4) admits a positive equilibrium point z_e, which is asymptotically stable, thus $\mathcal{D}_{sa}(z_e) \neq \emptyset$ and is supposed to satisfy $\{z \in \mathbb{R}_+^k : z \leq z_e\} \subsetneq \mathcal{D}_{as}(z_e) \subset S$,

3) $S_{SC} \subset \mathcal{D}_{as}(z_e)$ and $sc > z_e$ such that $g^K(sc) < 0$,

4) $(S_{SC} \cap S_{SC}(\hat{K})) = S_{SC}$.

Then with $\mathcal{A} = \{x \in \mathbb{R}^n : p(x) \leq z_e\}$, the problem $\mathcal{P}((LSF)\ (1), \mathcal{A}, S_{SC}, S_{SC}(K))$ has a solution $\hat{K}$. ■

Proof

Hypotheses 1) to 4) imply, using lemma 4, that $(S_{SC} \cap S_{SC}(\hat{K})) = S_{SC}$ is positively invariant w.r.t. $\{(LSF)\ (1), S_D\}$, thus $\forall\, d \in S_D$, $\forall\, t_0 \in \mathbb{R}$, $\forall\, x_0 \in S_{SC}$ property P1) of the problem $\mathcal{P}$ is satisified.

Let us suppose that property P2) of the problem $\mathcal{P}((LSF)\ (1), \mathcal{A}, S_{SC}, S_{SC}(K))$ is not satisified. Thus there exist $(d, t_0, x_0) \in S_D \times \mathbb{R} \times S_{SC}$ and a neighbourhood $\mathcal{N}(\mathcal{A})$ of $\mathcal{A}$ such that the solution of (LSF) (1) $\underline{x}(t\ ;\ t_0, x_0\ ;\ d\ ;\ \hat{K})$ lies out of $\mathcal{N}(\mathcal{A})$ for every $t \in [t_0, +\infty[$. And thus, using Lemma 1, we can exhibit a solution of (4):

$\underline{z}(t\ ;\ t_0, z_0 = p(x_0);\ \hat{K}) \geq p[\underline{x}(t\ ;\ t_0, x_0\ ;\ d\ ;\ \hat{K})]$ lying out of a neighbourhood of z_e, which is in contradiction with the attractivity hypothesis of z_e (hypothesis 2).

Remarks 5:

1) Strong and precise results concern the study of non linear time invariant systems such as system (4) (see [10, 11, 18]), then condition 2) of theorem 1 or lemma 4 can be worked out in practice.

2) Conditions 2), 3) and 4) of theorem 1 or lemma 4 are geometric and easy to test, whereas condition 1) can be tested using the property P2'' of lemma 3.

3) There exist techniques providing g^K of the following particular form (see [4, 5, 21]): $g^K = M\,z + q$, where M is the opposite of an M-matrix and q is a positive vector.

And thus, from theorem 1, one can obtain the following result:

Corollary 1

Let us suppose that there exists $\hat{K}$ such that:

1) the V.N. leads to (3) with $g^K = M\,z + q$, where M is a −M-matrix and q is a positive vector,

2) $S_{SC} \subset \mathcal{S}$ (the domain of validity of (3)),

3) $(S_{SC} \cap S_{SC}(\hat{K})) = S_{SC}$.

Then with $\mathcal{A} = \{x \in \mathbb{R}^n : p(x) \leq -M^{-1}q\}$, the problem $\mathcal{P}((LSF)\ (1), \mathcal{A}, S_{SC}, S_{SC}(K))$ has a solution $\hat{K}$. ■

Proof

It comes directly from Theorem 1.

Remark 6:

This result was used implicitly in [5] and in [20, 21, 22].

4.3. Methodology

The following algorithm permits, for a given system (*LSF*) (1) with given sets S_D, S_{SC}, S_{CC}, to find a gain matrix K which minimizes a set $\mathcal{A}$ and answers problem $\mathcal{P}$.

Step 1:
Choose an adequate V.N $p(x)$ and use the results of section 3 (see also [4, 6, 8, 11]) to obtain Inequality System (3).

Step 2:
Using property P2'' of lemma 3, test the quasi-monotonicity of $g^K \in C^1(S, \mathbb{R}^k)$.

Step 3:
Check the positive equilibrium points z_e (these points are depending on the variable K) of the Overvaluing System (4). Using Kotelianski criterion for local asymptotical stability gives condition on K (see [4, 6, 8, 11]).

Moreover, we obtain other conditions on K ensuring that $(S_{SC} \cap S_{SC}(\hat{K})) = S_{SC}$ and that it is positively invariant.

Step 4:
Subject to these conditions, minimize the size of $\mathcal{A} = \{x \in \mathbb{R}^n : p(x) \leq z_e\}$.

If not possible, try to find the greatest S_I positively invariant and included in $S_{SC} \cap S_{SC}(K)$ such that the procedure is possible. Verify if all conditions of Theorem 1 are fulfiled and conclude.

4.4. Example

Let us consider the disturbed system described by:

$$\frac{dx}{dt} = \begin{bmatrix} 1+|x_2| & 2\sin t \\ \sin t & -2 \end{bmatrix} x + \begin{bmatrix} k\ x_1 \\ 0 \end{bmatrix} + d(t),\ t \in \mathbb{R},\ x \in \mathbb{R}^n, \tag{18}$$

with time continuous motions $\underline{x}(t\,;\,t_0, x_0)$, $|d_i(t)| \leq \delta$, and the following sets of constraints:

$$S_{SC} = \{x \in \mathbb{R}^2 : |x_1| \leq 1 \text{ and } |x_2| \leq 1\}, \tag{19}$$

$$S_{CC} = \{c = k\,x_1 : |c| \leq 10\}. \tag{20}$$

Step 1:
Let $p(x)$ be the V.N. defined by:

$$p(x) = \begin{bmatrix} |x_1| \\ |x_2| \end{bmatrix}. \tag{21}$$

This yields:

$$D_t^+ p(x) \le g^K(p(x)),\ p(x) \in \mathbb{R}^2, \text{ with:} \tag{22}$$

$$g^K(z) = \begin{bmatrix} 1+k+z_2 & 2 \\ 1 & -2 \end{bmatrix} z + \begin{bmatrix} \delta \\ \delta \end{bmatrix}. \tag{23}$$

Step 2:
$g^K(z) \in C^1(\mathbb{R}^2, \mathbb{R}^2)$ is quasi-monotone non-decreasing.

Step 3:
The study of system $\frac{dz}{dt} = g^K(z)$ shows that there are two equilibrium points. There is only one equilibrium point z_e which belongs to S_{SC} (19) if $k < -2$, and it is given by:

$$s(k, \delta) = \sqrt{4\,(2+k)^2 - 4\,(2-k)\,\delta + \delta^2},$$

$$z_{1e} = 2\,z_2 - \delta = \frac{1}{2}\,\{-\delta - 2\,(2+k) - s(k, \delta)\}, \tag{24}$$

$$z_{2e} = \frac{1}{4}\,\{\delta - 2\,(2+k) - s(k, \delta)\}. \tag{25}$$

Moreover this point is (globally) asymptotically stable if:

$$\delta + 2k - s(k, \delta) < 0. \tag{26}$$

Step 4:
$(S_{SC} \cap S_{SC}(\hat{K})) = S_{SC}$ if:

$$|k| \le 10, \tag{27}$$

and it is invariant if:

$$k + 4 + \delta < 0, \tag{28}$$

$$-1 + \delta < 0, \tag{29}$$

To minimize the surface of the final taget set $\mathcal{A}$ defined by the equilibrium point z_e, we have to minimize $z_{1e}\, z_{2e}$ which depends on k under the constraints (26)-(29). And the study of the function:

$$h(k) = z_{1e}\, z_{2e} = \delta k - \frac{1}{2}(2+k)\ \{(\delta - 2\,(2+k)) - s(k,\,\delta)\}, \tag{30}$$

shows that it is strictly increasing when $-10 \le k \le -4$ (given by (27)-(29)). Thus the final set A is minimized for $k = -10$, and conditions (26)-(28) are satisfied.

Finally, using Theorem 1, we conclude that the problem $\mathcal{P}$((18), $\mathcal{A}$, S_{SC} (19), $S_{SC}(K)$ (20)) has a solution $\hat{K} = -10$, with:

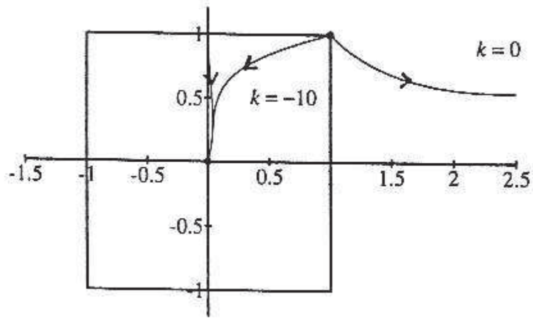
$$\mathcal{A} = \{x \in \mathbb{R}^n : p(x) \le \begin{bmatrix} \dfrac{16-\delta-\sqrt{(16+\delta)^2-80\delta}}{2} \\ \dfrac{16+\delta-\sqrt{(16+\delta)^2-80\delta}}{4} \end{bmatrix}\},\ 0 < \delta < 1. \tag{31}$$

If we suppose that δ is very small, then neglecting second and higher terms we obtain:

$$\mathcal{A} = \{x \in \mathbb{R}^n : p(x) \le \delta\left[\frac{1}{4},\ \frac{5}{8}\right]^{T}\},\ 0 < \delta << 1. \tag{32}$$

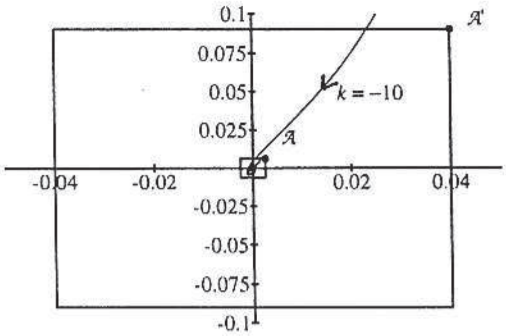
Using a linear O.S as in the paper of Radhy [20, 21], we obtain a final target set $\mathcal{A}'$, fourteen times bigger than $\mathcal{A}$:

$$\mathcal{A}' = \{x \in \mathbb{R}^2 : p(x) \le \delta\,[4,\ 9]^{T}\}. \tag{33}$$



-Figure 1: A solution to $\mathcal{P}$((18), $\mathcal{A}$, S_{SC} (19), $S_{SC}(K)$ (20))-

The computation of the solutions has been obtained with $d_1(t) = d_2(t) = 10^{-2}\cos(t)$. Figure 1 shows that the open loop system ($k = 0$) diverges, whereas it converges asymptotically and within the constaints, to the final taget A with $k = -10$. Figure 1 compares the obtained final target $\mathcal{A}$ with $\mathcal{A}'$ obtained using the method proposed by Radhy [20, 21].



-Figure 2: Comparison with the method proposed in Radhy [20, 21]-

If there is no perturbation, $\mathcal{A} = \{O\}$, the system converges to the origin $\{O\}$. Moreover the size of the final target set is directly linked to the size of the perturbation. We can also notice that if the input is constrained to a value smaller than the one used in (20), one can have a similar reasonning.

5. CONCLUSION

The original contributions of this paper are:

- lemma 3 connecting Wazewski's conditions with opposite of M-matrices, thus numerous results concerning –M-matrices (for example see [1, 3, 8, 11]) can be extended to more general O.S. with quasi-monotone functions,
- lemma 4 giving a simple condition for set $\mathcal{X}_w$ to be positively invariant, or with a slight modification (see remark 4), conditions for a set to be an estimate of the domain of asymptotic stability,
- theorem 1 giving sufficient conditions for the existence of a solution to the generalized constrained control problem $\mathcal{P}$, and a method that provides this solution.

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