Dynamic Systems and Applications 2 (1993) 405-404

STABILITY ANALYSIS OF TIME-DELAY SYSTEMS

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ABSTRACT: This paper presents a new way of studying stability of nonlinear delayed systems, by using vector norms and comparison principle. This allows to apply, with some modifications, the stability criteria available for linear systems. Examples illustrate the method.

AMS (MOS) Subject Classification: 34K20, 34K25, 34A40

1. INTRODUCTION

The modelling of numerous processes involves *time-delay differential equations*, taking into account the phenomenon of delay or post-effect. In free motion, such equations are described by :

$$\text{(E)} \qquad \frac{dx(t)}{dt} = \dot{x}(t) = f(t, x(t), x(t-\tau_1), ..., x(t-\tau_k)),$$

where t is the time, x a vector of $\mathbb{R}^n$, and $0 < \tau_1 < ... < \tau_k$ are constant parameters. Obviously, k = 0 is the "ordinary" case, that will not be considered here. When k = 1, the system (E) is said to be *with single delay* and when the τ_i are all integer multiples of a same real τ (called basic delay), the system (E) is said to be *with commensurate delays*. These equations belong to the more general class of *functional-differential equations*.

Time-delay equations have been discussed in literature since the early 1700's, beginning with Bernoulli and Euler. However, their works dealt with the study of special properties for particular equations. In his research on predator-prey models and viscoelasticity, Volterra ([29] and [30]) was one of the first to develop a more detailed theory for functional-differential equations. In recent decades, the theory of time-delay systems has experienced a rapid development and since the 50's a few books have been available on the subject (see for example Mishkis [18], Bellman and Cooke [2], El'sgol'ts [5], Halanay [7], Hale [8]).

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The fields of applications are various : in theory of numbers, the equation

$$\dot{x}(t) = -\alpha\, x(t-1)\,[1 + x(t)]$$

appears in studies of the distribution of primes (cf Wright [32]). In biology, Bailey and Reeve [1] have encountered retarded differential-difference equations in their study of the distribution of labelled albumin in the human blood system. Such systems are also common in the theory of epidemics. Fields of electronics, economics, nuclear and automatic control are also concerned. For this latter, Minorsky was one of the first scientists, in his study of ship stabilization and automatic steering [17], to show the importance of the delay in feedback mechanisms.

Vector x of the equation (E) is defined by several authors as the *instantaneous state* of the system, the *state* of the system at instant t is then the trajectory segment $x_t(\theta) = x(t+\theta)$ for $-\tau_k \leq \theta \leq 0$ (Shimanov's notation [25]), and one defines on this state space the norm : $\| x_t \| = \sup_{\theta \in [-\tau_k, 0]} | x(t+\theta) |$ where $| . |$ is a given norm of $\mathbb{R}^n$.

Initial values are determined by the knowledge of the initial instant t_0 and of n functions $\varphi_i(t)$, defined and continuous on $[-\tau_k, 0]$, what we note with $\varphi(\theta) = [\varphi_1(\theta), ..., \varphi_n(\theta)]^T$ by $x_{t_0}(\theta) = \varphi(\theta)$, $\forall\, \theta \in [-\tau_k, 0]$.
It is assumed in the following that the function f has the sufficient properties for the existence of a unique solution defined and continuous on $[t_0, +\infty[$ for any initial value function $\varphi(\theta)$ (for instance, f satisfies a Lipschitz condition in its second argument [5]). We do the supplementary hypothesis that $f(t, 0, ..., 0) = 0$, so that $x = 0$ is a solution of (E).

Simulation of these systems is tricky. A first way is the step-by-step method which consists in getting by a classical method the numerical values of the solution of

$$\dot{x}(t) = f(t, x(t), \tilde{x}(t-\tau_1), ..., \tilde{x}(t-\tau_k))$$

where $\tilde{x}(t)$ is some polynomial interpolation of the discrete approximation of x on the interval already covered. But because of the integration approximation results of the interpolation, this method is not very precise. Another way of operating, proposed by Virk [28], consists in adapting the Runge-Kutta method for delay-differential systems.

As for ordinary differential equations, the stability analysis is a crucial step in the control synthesis of time-delay systems. Lyapunov's stability notion can be extended to delayed systems :
- The solution x=0 is called *stable* if for any $\varepsilon > 0$ and any $t_0 \in \mathbb{R}$, there is a $\delta = \delta(\varepsilon, t_0) > 0$ such that all continuous solutions of (E) for which $\| x_{t_0} \| \leq \delta$ holds, satisfy : $| x(t) | < \varepsilon$ for every $t \geq t_0$. Stability will be said *global* (or *in the whole*) if and only if the maximal $\delta(\varepsilon, t_0)$ of the preceding definition tends to $+\infty$ when $\varepsilon \rightarrow +\infty$.

- The solution $x = 0$ is *attractive* if there is a $\eta > 0$ such that every solution $x(t)$ of (E) for which $\| x_{t_0} \| \leq \eta$ holds, satisfies $\lim_{t \to +\infty} x(t) = 0$. Attractivity is said *global* when the preceding condition holds for $\eta = + \infty$.
- The solution $x = 0$ is *asymptotically stable* if it is both stable and attractive, and it is *globally asymptotically stable* if it is both stable and globally attractive.
- The solution $x = 0$ of a system with commensurate delays is *asymptotically stable independent of delay* if it is asymptotically stable for any value of the basic delay τ.

Lyapunov's second method is still valid, but with some modifications : for theoretic reasons, N. N. Krasovskii [14] showed that the use of functionals is a convenient choice because the state space is of infinite dimension. Nevertheless, choosing a classical Lyapunov function is not useless but needs to restrict the candidates to belong to a class of particular solutions (cf Razumikhin [23] and Lakshmikantham [15]).

Concerning the first Lyapunov's method, there exists an analogue to the Poincaré-Lyapunov's theorem asserting that asymptotic stability of the linearized system proves the asymptotic stability of the initial system [14], and this justifies the research of stability criteria for delayed systems with constant coefficients.

The next part (section 2) is a brief survey of the methods available for such linear delayed systems.

Section 3 presents an original way of comparing nonlinear behaviours to linear ones, which leads to global stability results. It is based on the notions of vector norms (VN) and overvaluing systems. This allows to apply criteria of linear systems to nonlinear ones, as an example will illustrate.

2. A BRIEF SURVEY OF CRITERIA FOR LINEAR SYSTEMS

Consider the class of linear systems

$$\text{(L)} \qquad \dot{x}(t) = A\,x(t) + \sum_{i=1}^{k} B_i\, x(t-\tau_i)$$

where A and B_i are real $n \times n$ matrices, and τ_i non-negative numbers. Such a linear system is said to be asymptotically stable if its unique equilibrium $x = 0$ is asymptotically stable. We know that if the roots (and there are, generally, an infinity) of the characteristic equation

$$p(s, \tau_1, \ldots, \tau_k) = \det \left[s\,I - A - \sum_{i=1}^{k} B_i\, e^{-\tau_i s} \right] = 0,$$

(obtained by searching the particular solutions of the exponential type : $e^{\lambda t} c$, where c is a constant vector) are with negative real parts then the system is asymptotically stable. This result is exactly the same that for ordinary linear systems but in the delayed case there is no criterion as simple as the Routh-Hurwitz test.

The criteria available for linear, autonomous systems can be classified in 3 groups :

- *algebraic methods* (state-space representation): on the basis of matrix measures, Mori proposed a basic criterion [19] which opened a serial of more easy-to-check corollaries [21, 22]. The Lyapunov's equation [4] and the concept of M-matrices [16, 20, 27] are also to be concerned as algebraic methods.

- *roots locus type methods* (frequency domain) consists in determination of the delay values that change the stability of the systems. Among them, the method of Walton and Marshall [31] is a powerful approach of linear systems with single or commensurate delays (see also [10]). Several authors proposed to replace $e^{s\tau}$ in the characteristic equation by rational fractions of a so-called pseudo-delay [9, 26].
- *polynomial methods* (frequency domain) globally consist in associating a bivariate (or multivariate) polynomial to the characteristic equation [11, 12, 13].

3. APPLICATIONS TO NONLINEAR DELAY-DIFFERENTIAL SYSTEMS

In order to analyse stability of a nonlinear system it is sometimes convenient to define a comparison system, this is, a system whose stability properties are characteristic of the original motion, and are simpler to investigate. Vector norms approach represents a practical way of defining comparison systems. Using this concept, Borne and Gentina [3] (see also [6] and [24]) have established stability criteria for nonlinear time-varying systems.
We propose now to extend the use of vector norm to nonlinear delay-differential systems, and more particularly for the construction of comparison systems.

3.1. Overvaluing systems and vector norms

3.1.1 Concept of regular vector norm

Consider the following partition of $\mathbb{R}^n$:

$$\mathbb{R}^n = E_1 \oplus E_2 \oplus \dots \oplus E_k ,$$

where $\oplus$ denotes the direct vector subspaces sum.
Let x be a vector of $\mathbb{R}^n$ with a projection in the subspace E_i denoted by x_i :

$$x_i = P_i\, x, \qquad \text{where } P_i \text{ is the projection operator from } \mathbb{R}^n \text{ into } E_i.$$

Let p_i be a (scalar) norm defined on the subspace E_i and p, which components are :

$$p_i(x) = p_i(x_i).$$

Then $p = (p_i)$ is a regular vector norm (VN) of dimension k :

$$p : \mathbb{R}^n \to \mathbb{R}_+^k$$

3.1.2 Overvaluing systems

We consider now the class of nonlinear time-varying systems described by the following vector delay-differential equation :

$$\text{(NL)} \qquad \dot{x}(t) = A(t, x(t), x(t-\tau))\, x(t) + B(t, x(t), x(t-\tau))\, x(t-\tau),$$

where $x \in \mathbb{R}^n$ is the instantaneous state vector, $\tau > 0$ is the delay, A(.) and B(.) represent $n \times n$ matrices :

$$A \text{ (or } B): \quad \mathcal{T}_0 \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{n\times n} , \text{ where } \mathcal{T}_0 = [t_0 , +\infty[, \ t_0 \in \mathbb{R}.$$

We assume that the system (NL) possess the adequate smoothness conditions ensuring the existence of continuous solutions $x(t, t_0, \varphi)$ for every $t_0 \in \mathbb{R}$ and for every initial function $\varphi(t)$ that is defined and continuous on $[t_0 - \tau, t_0]$.
The following definitions extend previous overvaluing systems [24] to the nonlinear delay-differential systems.
$D^+p_i(x_i)$ represents the right-hand Dini derivative of $p_i(x_i)$, taken along the motion of (NL), and we shall abridge $M(t, x(t), x(t-\tau))$ into $M(.)$.

Definition : The matrices $M, N : \mathcal{T}_0 \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{k\times k}$ define a delay-overvaluing system of (NL) with respect to the VN p if and only if the following inequality is satisfied along every motion of (NL) and for each corresponding component :

$$D^+p(x(t)) \le M(t, x(t), x(t-\tau))\, p(x(t)) + N(t, x(t), x(t-\tau))\, p(x(t-\tau)),$$
$$\forall\, t \in \mathcal{T}_0, \forall\, \varphi \in C([t_0 - \tau, t_0], \mathbb{R}^n),$$

where $M(.) = \{\mu_{ij}(.)\}$ is such that its off-diagonal elements are non-negative, and $N(.) = \{\nu_{ij}(.)\}$ is non-negative.

3.1.3 Application to usual Hölder's norms

For usual norms, it is easy to make explicit the natural system of delay-overvaluing matrices of a system (NL).
Let a partition of $\mathbb{R}^n$ define a block partition of matrix A. I_i and I_j represent the sets of indices of rows and columns, respectively, of block A_{ij}.
If $p_i(x_i)$ is the Euclidean norm of x_i, then denoting by $a_{ij}(.)$ and $b_{ij}(.)$ (for $i, j \in \{1, ..., n\}$) the elements of matrices $A(.)$ and $B(.)$, we obtain :

$$\mu_{ii}(.) = \max_{s \in I_i} \left[a_{ss} + \frac{1}{2} \sum_{\ell \in I_i} |a_{s\ell} + a_{\ell s}|\right], \quad \forall\, i \in \{1, ..., k\}$$

$$\mu_{ij}(.) = \frac{1}{2}\left(\max_{s \in I_i} \left[\sum_{\ell \in I_j} |a_{s\ell}|\right] + \max_{\ell \in I_j} \left[\sum_{s \in I_i} |a_{\ell s}|\right]\right), \quad \forall\, i \ne j \in \{1, ..., k\}$$

$$\nu_{ij}(.) = \frac{1}{2}\left(\max_{s \in I_i} \left[\sum_{\ell \in I_j} |b_{s\ell}|\right] + \max_{\ell \in I_j} \left[\sum_{s \in I_i} |b_{\ell s}|\right]\right), \quad \forall\, i \ne j \in \{1, ..., k\}$$

for all x in $\mathbb{R}^n$ and t in $\mathcal{T}_0$.
If $p_i(x_i)$ is the "max" norm (maximum of the modulus of each component of x_i), then :

$$\mu_{ii}(.) = \max_{s \in I_i} \left[a_{ss} + \sum_{\ell \in I_i - \{s\}} |a_{s\ell}|\right], \quad \forall\, i \in \{1, ..., k\}$$

$$\mu_{ij}(.) = \max_{s \in I_i} \left[\sum_{\ell \in I_j} |a_{s\ell}|\right], \quad \forall\, i \ne j \in \{1, ..., k\}$$

$$\nu_{ij}(.) = \max_{s \in I_i} \left[\sum_{\ell \in I_j} |b_{s\ell}|\right], \quad \forall\, i, j \in \{1, ..., k\}.$$

The dual norm of the max norm (i.e the modulus norm) leads to equations analogous to the precedent ones but inverting $\ell \in I_j$ and $s \in I_i$.

3.2. Application to stability analysis

3.2.1. Vector norms and comparison principle

The following lemma is a generalization of [6] to delay systems.

Comparison Lemma : Let there exist a VN p and two matrices M(.) and N(.) connected with (NL) such that the off-diagonal elements of M(.) and all the elements of N(.) are all non-negative and assume that the following inequality is satisfied along a solution of (NL) :

$$D^+p(x(t)) \leq M(.)\, p(x(t)) + N(.)\, p(x(t-\tau)),\ \ \forall\, t \in \mathcal{T}_0,\ \forall\, \varphi \in C([t_0-\tau, t_0], \mathbb{R}^n),$$

and suppose that system (C) :

$$\text{(C)} \qquad \dot{z}(t) = M(.)\, z(t) + N(.)\, z(t-\tau)$$

has time-continuous solutions.

Then (C) is a comparison system of (NL) in the sense that :

$$z(t) \geq p(x(t))\ ,\ \forall\, t \in \mathcal{T}_0$$

holds as soon as $z(s) \geq p(\varphi(s))$, for all s in the initial interval $[t_0 - \tau\ ;\ t_0]$.

Proof : ε denotes an error vector defined as follows : $\varepsilon(t) = z(t) - p(x(t))$, $\forall\, t \in \mathcal{T}_0$ where $x(t) = x(t, t_0, \varphi)$ as usual.

By assumption, for all s in $[t_0-\tau\ ;\ t_0]$, we have $\varepsilon(s) \geq 0$. Definitions of z and p involve :

$$D^+\varepsilon(t) \geq M(.)\, \varepsilon(t) + N(.)\, \varepsilon(t-\tau).$$

x and z are continuous, therefore ε is a continuous vector, so let us assume that $t_1 \geq t_0$ is the first moment when a component of ε (denoted by ε_i) becomes zero. Then, we can write:

$$D^+\varepsilon_i(t) \geq \sum_{\substack{j=1 \\ j \neq i}}^{k} \mu_{ij}(.)\, \varepsilon_j(t_1) + \sum_{j=1}^{k} \nu_{ij}(.)\, \varepsilon_j(t_1-\tau),$$

each term of the right member of this relation is non-negative, thus :

$$D^+\varepsilon_i(t) \geq 0.$$

So ε_i is a non-decreasing function at time t_1.

In conclusion, $\varepsilon(t)$ will remain non-negative $\forall\, t \geq t_0$, hence $z(t) \geq p(x(t))$ for all t in $\mathcal{T}_0$.

3.2.2 Practical Criteria of stability

Theorem 1 : The (asymptotic) stability of the solution x = 0 of a comparison system deduced from (NL) involves the (asymptotic) stability of the null solution of the initial system (NL).

Proof : This is obvious from the comparison lemma.

In particular, if (M, N) is a pair of constant matrices then results of Tokumaru *et al* [27] allows us to state the following theorem :

Theorem 2 : (i) If for the system (NL), it is possible to define for all $(t, x(t), x(t-\tau))$ in $\mathcal{T}_0 \times \mathcal{D} \times \mathcal{D}$ (where $\mathcal{D}$ is a region of $\mathbb{R}^n$ containing a neighbourhood of the origin) a constant delay-overvaluing system (M, N) related to a VN p with the additional property that M + N is the opposite of a M-matrix then the solution x = 0 of (NL) is asymptotically stable.
(ii) If (i) is valid for $\mathcal{D} = \mathbb{R}^n$, then the system (NL) is globally asymptotically stable.

Remark : This theorem is still valid for independent time-varying delays if we use the notations :

$$\tau(t) = [\tau_1(t), \tau_2(t), ..., \tau_n(t)]^T,$$
$$x(t-\tau(t)) = [x_1(t-\tau_1(t)), x_2(t-\tau_2(t)), ..., x_n(t-\tau_n(t))]^T,$$

and if we assume that all delays are sectionally smooth and bounded, i.e. :

$$0 \le \tau_i(t) \le \tau_0, \quad \forall\, t \in \mathcal{T}_0.$$

Theorem 3 : (i) If for the system (NL), it is possible to define for all $(t, x(t), x(t-\tau))$ in $\mathcal{T}_0 \times \mathcal{D} \times \mathcal{D}$ (where $\mathcal{D}$ is a region of $\mathbb{R}^n$ containing a neighbourhood of the origin) a constant delay-overvaluing system (M, N) related to a VN p such that the comparison system (C) is asymptotically stable (independent of delay or not) then the solution x = 0 of system (NL) is asymptotically stable.
(ii) If $\mathcal{D} = \mathbb{R}^n$, then the system (NL) is globally asymptotically stable.

3.3. Examples

3.3.1 Example 1

Consider the following system :

$$\text{(NL1)}\quad \dot{x}(t) = \begin{bmatrix} -4 & \text{sign}(x_2) \\ \text{sign}(x_1) & -4 \end{bmatrix} x(t) - \begin{bmatrix} 0 & 0 \\ 0 & a\sin(t) \end{bmatrix} x(t-2), \quad \text{for } t \ge 0,$$

$$\text{with}\quad \text{sign}(\alpha) = \begin{cases} 1 & \text{if } \alpha > 0 \\ 0 & \text{if } \alpha = 0 \\ -1 & \text{if } \alpha < 0 \end{cases}$$

The use of the regular vector norm $p_1(x) = [\,|x_1|, \,|x_2|\,]^T$ gives the following expression of the natural delay-overvaluing systems :

$$\text{(C1)}\quad \dot{z}(t) = \begin{bmatrix} -4 & |\text{sign}(x_2)| \\ |\text{sign}(x_1)| & -4 \end{bmatrix} z(t) + \begin{bmatrix} 0 & 0 \\ 0 & |a|\;|\sin(t)| \end{bmatrix} z(t-2).$$

We note here that for vector norms of form $p(x) = [\,|x_1|, ..., |x_k|\,]^T$, the expression of the natural delay-overvaluing system is easily obtained by taking the absolute values of all off-diagonal entries of the first matrix, and the absolute values of all entries of the second matrix.

By a new overvaluation of system (C1), we obtain the linear delay-overvaluing system of (NL1) with respect to p_1 :

$$\dot{y}(t) = \begin{bmatrix} -4 & 1 \\ 1 & -4 \end{bmatrix} y(t) + \begin{bmatrix} 0 & 0 \\ 0 & |a| \end{bmatrix} y(t-2).$$

Theorem 2 shows then that system (NL1) is asymptotically stable for $|a| < 3{,}75$.

3.3.2 Example 2

We consider the following system (NL2):

$$\dot{x}(t) = \begin{bmatrix} -4 & \sin x_3 & 0{,}2 \\ -0{,}1 & -3 & 0{,}6 \sin x_1 \\ \sin t & 0{,}5 \sin x_2(t-1) & -3 \end{bmatrix} x(t) + \begin{bmatrix} 0{,}2 & 0{,}3 & \sin x_3 \\ \sin x_1(t-1) & 0{,}1 & 0{,}5 \\ 0{,}1 & 0{,}1 \sin t & 1 \end{bmatrix} x(t-1).$$

The use of the regular VN $p(x) = [\max(|x_1|, |x_2|), |x_3|]^T$
induces the linear comparison system :

(C2) $$\dot{z}(t) = \begin{bmatrix} -2{,}9 & 0{,}6 \\ 1{,}5 & -3 \end{bmatrix} z(t) + \begin{bmatrix} 1{,}1 & 1 \\ 0{,}2 & 1 \end{bmatrix} z(t-1).$$

We easily verify that the matrix :

$$\begin{bmatrix} -1{,}8 & 1{,}6 \\ 1{,}7 & -2 \end{bmatrix}$$

is the opposite of a M-matrix, so asymptotic stability of (NL2) is proved by theorem 2.

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