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STABILITY AND STABILITY DOMAINS ANALYSIS FOR NONLINEAR DIFFERENTIAL-DIFFERENCE EQUATIONS

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ABSTRACT: In this paper, the problem of stability analysis of nonlinear time-delay systems is considered by using comparison-like methods. The main tool is the concept of vector norms which gives a systematic way of defining comparison systems. Examples illustrate the study.

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1. INTRODUCTION

Comparison principle for functional differential equations [8], [12] coupled together with the vector Lyapunov functions approach [7], [10] is an efficient way of analysing stability of complex time-delay systems (see for instance [2], [13], or [14]).

In reference [2] the authors provided an approach based on vector norms and comparison systems, by adapting previous results obtained in the non-delayed case ([1], [6], [11]) to delay-systems. Following this method, this paper investigates delay-independent stability properties of a general class of nonlinear, time-varying delay-differential systems described by :

$$\dot{x}(t) = A(t, x(t), x(t-\tau(t)))\, x(t) + B(t, x(t), x(t-\tau(t)))\, x(t-\tau(t)), \qquad (1)$$

where $x \in \mathbb{R}^n$ is the instantaneous state vector, $A(.)$ and $B(.)$ represent $n \times n$ matrices, τ is a piecewise continuous function of the variable t satisfying $0 \le \tau(t) \le \tau_0$,
and it is assumed that there is a unique solution $x(t, t_0, \varphi)$ of (1) for every $t_0 \in \mathbb{R}$ and for every initial vector function $\varphi(t)$ defined and continuous on $[t_0 - \tau_0, t_0]$.

This paper presents two main contributions:
- the first original point is to provide estimates of the attraction domain linked to an asymptotically stable solution. This is obtained by using comparison systems which can be valid only locally.
- the second one is to provide stability criteria that can be directly applied to nonlinear, delayed systems. This allows us to enlarge the class of systems that can be analysed, as will be shown in example 2, section 4.

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2. COMPUTATION OF COMPARISON SYSTEMS

We first recall the vector norm concept [11]:
Consider the following partition of $\mathbb{R}^n$: $\mathbb{R}^n = E_1 \oplus E_2 \oplus \ldots \oplus E_k$, where $\oplus$ denotes the direct vector subspaces sum. Let P_i be the projection operator from $\mathbb{R}^n$ onto E_i, and x be a vector of $\mathbb{R}^n$. The projection of x onto E_i will be denoted by x_i, so $x_i = P_i\, x = P_i\, x_i$.
Let p_i be a norm on the subspace E_i, $(i = 1, \ldots, k)$. Then the vector function $p : \mathbb{R}^n \to \mathbb{R}^k_+$, whose i^{th} - component is defined by $p_i(x) = p_i(x_i)$, is a *regular vector norm* (VN) of dimension k.

In the sequel, the following notations and conventions are used: $D^+p_i(x_i)$ represents the right-hand derivative of $p_i(x_i)$ with respect to time taken along the motions of (1). $\mathcal{T}_0$ denotes the interval $[t_0, +\infty[$. $\mathcal{D}$ is a region of $\mathbb{R}^n$ containing a neighbourhood of the origin. $C = C([-\tau_0, 0], \mathbb{R}^n)$ is the set of continuous functions that map the interval $[-\tau_0, 0]$ into $\mathbb{R}^n$. $x_t \in C$ is defined by $x_t(s) = x(t+s)$, $-\tau_0 \le s \le 0$. In the third section, $\mathfrak{I}_u(\alpha)$ will denote the set $\{\varphi \in C : p(\varphi(s)) \le \alpha\, u,\ \forall\, s \in [-\tau_0, 0]\}$ and $I_u(\alpha)$ the set $\{x \in \mathbb{R}^n : p(x) \le \alpha\, u\}$, where u is a given vector, and α a positive number. Any vector or matrix inequality $A \le B$ is to be understood component-by-component, and $M(t, x(t), x(t-\tau(t)))$ will be abbreviated by $M(.)$.

Definition 1 : The matrices $M, N : \mathcal{T}_0 \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{k \times k}$ define an *overvaluing system* of (1) with respect to the VN p and the region $\mathcal{D}$ if the following inequality is satisfied along every motion of (1):

$$D^+p(x(t)) \le M(t, x(t), x(t-\tau(t)))\, p(x(t)) + N(t, x(t), x(t-\tau(t)))\, p(x(t-\tau(t))), \quad (2)$$
$$\forall\, t \in \mathcal{T}_0,\ \forall\, x_t \in C([-\tau_0, 0], \mathcal{D}),$$

where $M(.) = \{\mu_{ij}(.)\}$, is such that its off-diagonal elements are non-negative, and $N(.) = \{\nu_{ij}(.)\}$ is a non-negative matrix.
If $\mathcal{D} = \mathbb{R}^n$, the overvaluing system is said to be global.

This definition is an extension of the ones developped in the non-delayed case (see for example [6], and [11]), and in [2] where the global case was the only considered. In the sequel, the following assumption will be used:

Assumption 1: The pair of matrices (M, N) is such that system

$$\dot{z}(t) = M(t, x(t), x(t-\tau(t)))\, z(t) + N(t, x(t), x(t-\tau(t)))\, z(t-\tau(t))$$

admits a unique solution for any $x_{t_0} \in C([-\tau_0, 0], \mathcal{D})$ and any $z_{t_0} \in C([-\tau_0, 0], p(\mathcal{D}))$. For instance, M(.) and N(.) may be locally Lipschitzian with respect to their second and third arguments (see [4])

It is possible to give, for every system of form (1), the expressions of a particular overvaluing system. Let :

$$y = x(t-\tau(t)),$$
$$A_{ij}(.) = P_i\, A(.)\, P_j\,, \quad B_{ij}(.) = P_i\, B(.)\, P_j$$
$$m_{ij}(t, x, y, u) = \frac{\text{grad}\ p_i(u_i)^T A_{ij}(.)\, u_j}{p_j(u_j)}\,, \quad (3)$$

$$n_{ij}(t, x, y, u, v) = \frac{\text{grad } p_i(u_i)^T B_{ij}(.) v_j}{p_j(v_j)}$$

Theorem 1 : The matrices M(t, x, y), N(t, x, y) given by :

$$\mu_{ij}(t, x, y) = \sup_{u \in \mathbb{R}^n} \{m_{ij}(t, x, y, u)\}$$

$$\nu_{ij}(t, x, y) = \sup_{u, v \in \mathbb{R}^n} \{n_{ij}(t, x, y, u, v)\}, \forall i, j \text{ and } \forall t \in \mathcal{T}_0, \forall x, y \in \mathbb{R}^n \quad (4)$$

define an overvaluing system of (1) with respect to the VN p. Systems given by (4) are called *natural overvaluing systems* of (1).

Proof : Using (1), the Dini derivative of p_i can be expressed by:

$$D^+ p_i(x) = \sum_{j=1}^{k} \text{grad } p_i(x_i)^T A_{ij}(.) x_j + \sum_{j=1}^{k} \text{grad } p_i(x_i)^T B_{ij}(.) y_j .$$

If $x_j \neq 0$, we have : $\text{grad } p_i(x_i)^T A_{ij}(.) x_j = \frac{\text{grad } p_i(x_i)^T A_{ij}(.) x_j}{p_j(x_j)} p_j(x_j)$,

and, by definition of $\mu_{ij}(t, x, y)$, it yields: $\text{grad } p_i(x_i)^T A_{ij}(.) x_j \leq \mu_{ij}(.) p_j(x_j)$.

If $x_j = 0$, the last inequality obviously holds. In the same way, it is possible to prove : $\text{grad } p_i(x_i)^T B_{ij}(.) y_j \leq \nu_{ij}(.) p_j(y_j)$. So, inequality (2) is proved.

Independence of vectors u_i and u_j and of vectors u_i and v_j in (3) ensures that off-diagonal entries of M(.) and all the elements of N(.) are non-negative.

Corollary 1 : Any matrices $\tilde{M}(.)$ and $\tilde{N}(.)$ such that :

$$\tilde{\mu}_{ij}(t, x, y)) \geq \mu_{ij}(t, x, y),$$

$$\tilde{\nu}_{ij}(t, x, y)) \geq \nu_{ij}(t, x, y),$$

$$\forall i, j = 1, \ldots, k \text{ and } \forall t, x, y \in \mathcal{T}_0 \times \mathbb{R}^n \times \mathbb{R}^n,$$

define an overvaluing system of (1).

The use of this corollary may provide simple forms of overvaluing systems, for example M and N may be constant.

For usual norms, expressions of the natural overvaluing system are given in an explicit form. Let a partition of the space $\mathbb{R}^n$ define a block partition of matrix A. I_i and I_j represent the sets of indices of rows and columns, respectively, of block A_{ij}.

With $p_i(x_i) = \| x_i \|$, where $\| . \|$ is an arbitrary norm on E_i, we obtain

$$\begin{aligned} \mu_{ii}(.) &= \mu(A_{ii}), \\ \mu_{ij}(.) &= \| A_{ij} \|, \text{ for } i, j = 1, \ldots, k \text{ and } i \neq j \\ \nu_{ij}(.) &= \| B_{ij} \|, \text{ for } i, j = 1, \ldots, k, \end{aligned} \quad (5)$$

where $\mu(A_{ii})$ is the matrix measure of A_{ii} associated with the norm $\| . \|$ (see [3]). These formulas generalize the ones given in [2].

We now propose an extension of Borne and Gentina's comparison lemma ([6]) to the nonlinear differential-difference equations.

Comparison lemma 1: Let (M(.), N(.)) define an overvaluing system of (1) with respect to a regular VN p and a region $\mathcal{D}$, and such that system

$$\dot{z}(t) = M(t, x(t), x(t-\tau(t)))\, z(t) + N(t, x(t), x(t-\tau(t)))\, z(t-\tau(t)) \quad (6)$$

satisfies Assumption 1.

Then (6) is a *comparison system* of (1) in the sense that if the inequality

$$z(t) \geq p(x(t)) \quad (7)$$

holds for $t \in [t_0-\tau_0, t_0]$, then it holds as long as x(t) remains in $\mathcal{D}$.

Proof : The proof of this lemma is a direct adaptation of Grujić *et al.* [6], and for sake of brevity is not reproduced here. ◊

Corollary 2: Stability (or respectively attractivity, asymptotic stability) of the zero solution of a comparison system (6) deduced from (1) implies stability (respectively attractivity, asymptotic stability) of the zero solution of system (1).

3. STABILITY THEOREMS AND STABILITY DOMAINS ESTIMATION

A classical result obtained by Tokumaru *et al.* [12] is that if (6) is linear, time-invariant (i.e. the matrices M(.) and N(.) are constant and verify conditions of definition 1) then it is asymptotically stable independent of delay if and only if the sum-matrix M+N is the opposite of an M-matrix (see Appendix). The following theorems 2 and 3 generalize this result to two classes of nonlinear, delayed systems (they can also be regarded as generalizations of Borne and Gentina's results). Theorem 4 is an extension of Mori's results [9].

In this first result, we consider that the delay $\tau(t)$ is constant, i.e. $\tau(t) = \tau_0$ for all t in $\mathcal{T}_0$.

Theorem 2 : If for the system (1) there is an overvaluing system:

$$\dot{z}(t) = M(t, x(t), x(t-\tau_0))\, z(t) + N(t, x(t), x(t-\tau_0))\, z(t-\tau_0),$$

related to a regular VN p and a region $\mathcal{D}$, satisfying Assumption 1, such that:

(i) non-constant elements of $Z_1(t, x, y, w) = M(t, x, y) + N(t+\tau_0, w, x)$ are isolated in one column, and

(ii) there is $\varepsilon > 0$ such that, for all t in $\mathcal{T}_0$ and all x, y, w in $\mathcal{D}$,

$Z_1(t, x, y, w) + \varepsilon I_k$ is the opposite of an irreducible M-matrix,

then the zero solution of (1) is stable (asymptotically stable if N(.) is bounded), and stability (or asymptotic stability) is global if $\mathcal{D} = \mathbb{R}^n$.

Proof : Let $\lambda_{max} = \sup[\lambda_m(Z_1(t, x, y, w));\ (t, x, y, w) \in \mathcal{T}_0 \times \mathcal{D}^3]$ (where $\lambda_m(Z_1)$ is the importance eigenvalue of Z_1, see Appendix), so $\lambda_{max} \leq -\varepsilon$, and let u be an eigenvector of $Z_1^T(t, x, y, w)$ associated with λ_{max} for a given (t_1, x_1, y_1, w_1).

Let $V(x_t) = p^T(x(t)).u + \int_{t-\tau_0}^{t} p^T(x(s)).N^T(s+\tau_0, x(s+\tau_0), x(s))u \, ds$

be a tentative Lyapunov-Krasovskii functional, then from (2) we obtain:

$$D^+V(x_t) \leq p^T(x(t)) [M(t, x(t), x(t-\tau_0)) + N(t+\tau_0, x(t+\tau_0), x(t))]^T u$$

Then, conditions (i), (ii), and lemma 2 (in Appendix) yield:

$[M(t, x(t), x(t-\tau_0)) + N(t+\tau_0, x(t+\tau_0), x(t))]^T u \leq \lambda_{max} u \leq -\varepsilon u,$

so $D^+V(x_t) \leq -\varepsilon \, p^T(x(t)).u$

According to Krasovskii's theorem, the solution $x = 0$ of (1) is stable, or asymptotically stable if N(.) is bounded.

Theorem 3 : If for the system (1), it is possible to define an overvaluing system

$$\dot{z}(t) = M(t, x(t), x(t-\tau(t)))\, z(t) + N(t, x(t), x(t-\tau(t)))\, z(t-\tau(t)),$$

related to a regular VN p and a region $\mathcal{D}$, satisfying Assumption 1, such that:

(i') non-constant elements of $Z_2(t, x, y) = M(t, x, y) + N(t, x, y)$ are isolated in one row,

(ii') there is $\varepsilon > 0$ such that, for all t in $\mathcal{T}_0$ and all x, y in $\mathcal{D}$,

$Z_2(t, x, y) + \varepsilon I_k$ is the opposite of an irreducible M-matrix,

then the solution $x = 0$ of (1) is stable (asymptotically stable if N(.) is bounded) and the set $\mathfrak{S}_{u_c}(\alpha)$, where $u_c = u_c(\mathcal{D})$ is an importance eigenvector related to $\lambda_{max} = \max\{\lambda_m(Z_2(t, x, y))\,;\, t \in \mathcal{T}_0, x, y \in \mathcal{D}\}$, and α is any positive real such that the set $I_{u_c}(\alpha)$ is included in $\mathcal{D}$, is a positively invariant set with respect to (1).

Proof : Let $v(x) = \text{Max}\left(\frac{p_1(x_1)}{u_{c,1}}, \ldots, \frac{p_k(x_k)}{u_{c,k}}\right)$

For any time t in $\mathcal{T}_0$, there is an index $i \in \{1, \ldots, k\}$ such that:

$$D^+v(x(t)) = \frac{1}{u_{c,i}} D^+p_i(x(t)),$$

so $D^+v(x(t)) \leq \frac{1}{u_{c,i}} [\mu_{ii}(.)\, p_i(x_i(t)) + \sum_{j\neq i} \mu_{ij}(.)\, p_j(x_j(t)) + \sum_{j=1}^{k} \nu_{ij}(.)\, p_j(x_j(t-\tau(t)))],$

then, from the definition of v(x), we deduce

$$D^+v(x(t)) \leq \frac{1}{u_{c,i}} [M(.)\, u_c\, v(x(t)) + N(.)\, u_c\, v(x(t-\tau(t)))]_i$$

Following Razumikhin's method, we only consider the solutions satisfying

$$v(x(t)) \leq v(x(t-\tau(t))).$$

Then, $D^+v(x(t)) \leq \lambda_{max}\, v(x(t)) < 0.$

Thus, v(x) is a Lyapunov-Razumikhin function, and hence, solution $x = 0$ is stable.

If N(.) is bounded, the solutions satisfying $v(x(t)) \leq (1+\alpha)\, v(x(t-\tau(t))$ for all t in $\mathcal{T}_0$, where α is a sufficiently small positive number, verify $D^+v(x(t)) < 0$, so according to [7] or [10], the solution $x = 0$ is asymptotically stable.

At last, the sets $\{x \in \mathbb{R}^n : v(x) \leq \alpha\}$ contained in $\mathcal{D}$ are positively invariant (see [7]).

Remark: When the matrices M(.) and N(.) of the overvaluing systems are constant, theorems 2 and 3 hold simultaneously : we encounter the classic condition given in Tokumaru *et al.* [12], but, in the case of a local comparison system ($\mathcal{D} \neq \mathbb{R}^n$), theorem 3

and the following corollary give the definition of positively invariant sets which estimate the stability domain of the null solution.

Corollary 3: If for the system (1), it is possible to define for all (t, x, y) in $\mathcal{T}_0 \times \mathcal{D} \times \mathcal{D}$ a linear, time-invariant overvaluing system $\dot{z}(t) = M\, z(t) + N\, z(t - \tau(t))$ (M, N constant matrices) related to a VN p with the additional property that M+N is the opposite of an irreducible M-matrix, then
i) the solution x = 0 of (1) is asymptotically stable (Tokumaru *et al.*'s criterion [12]), and
ii) an estimate of the attraction domain is given by the maximal set $\mathcal{S}_d(\alpha)$ for which α is such that $I_d(\alpha)$ is included in $\mathcal{D}$ and d is any vector such that $(M+N)d < 0$.
Moreover, if $\mathcal{D} = \mathbb{R}^n$, then the system (1) is globally asymptotically stable.

The use of one-dimensional vector norms permits the formulation of a theorem that extends Mori's results [9] to the nonlinear case.

Theorem 4 : If there are two functions a(t), b(t) such that :

(i) $\exists\, \varepsilon > 0, \forall\, t \in \mathcal{T}_0,\ a(t) \le -\varepsilon$,

(ii) $\forall\, t \in \mathcal{T}_0,\ a(t) + b(t) < 0$ and

(iii) $\forall\, t \in \mathcal{T}_0, \forall\, x, y \in \mathbb{R}^n,\ \mu(A(t, x, y)) \le a(t)$ and $\|B(t, x, y)\| \le b(t)$,

(iv) The system $\dot{z}(t) = a(t)\, z(t) + b(t)\, z(t - \tau(t))$ satisfies Assumption 1.

where $\|\,.\,\|$ denotes an induced matrix norm, and $\mu(.)$ is the corresponding matrix measure (see [3]), then the zero solution of (1) is asymptotically stable.

Proof : Consider $p(x) = \|x\|$ then :

$$D^+p(x(t)) = \lim_{h \to 0^+} h^{-1} \left[\|x(t+h)\| - \|x(t)\|\right]$$

$$= \lim_{h \to 0^+} h^{-1} \left[\|x(t) + h(A(.)x(t) + B(.)x(t - \tau(t)))\| - \|x(t)\|\right]$$

So :

$$D^+p(x(t)) \le \lim_{h \to 0^+} h^{-1} \left[\|I + h\, A(.)\| - 1\right] \|x(t)\| + \|B(.)\|\, \|x(t - \tau(t))\|$$

By definition of $\mu(A(.))$, we have

$$D^+p(x(t)) \le \mu(A(.))\, p(x(t)) + \|B(.)\|\, p(x(t - \tau(t)))$$

Thus,

$$\dot{z}(t) = a(t)\, z(t) + b(t)\, z(t - \tau(t))$$

is a comparison system deduced from (1). Then, using Razumikhin's method [10], with the Lyapunov function $V(z) = z^2$, it is easy to show that, under conditions (i) and (ii), the solution x = 0 of (1) is asymptotically stable. ◊

4. EXAMPLES

Example 1: Let us consider the system described by the relation:

$$\dot{x}(t) = \begin{bmatrix} -3 + x_1 & \sin y_2 \\ 2\cos t & -4 + y_1 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ 0 & \sin x_1 \end{bmatrix} x(t-2). \qquad (8)$$

where $y(t) = x(t-2) = [y_1, y_2]^T$.

Let p be defined as: $p(x) = [|x_1|, |x_2|]^T$ (9)
therefore:

$$M(t, x, y) = \begin{bmatrix} -3 + x_1 & |\sin y_2| \\ 2\,|\cos t| & -4 + y_1 \end{bmatrix} \leq \begin{bmatrix} -3 + x_1 & 1 \\ 2 & -4 + y_1 \end{bmatrix}$$

$$\text{and } N(t, x, y) = \begin{bmatrix} 0 & 0 \\ 0 & |\sin x_1| \end{bmatrix} \leq \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = N.$$

On the domain $\mathcal{D}(\varepsilon) = (-\infty, 3 - \sqrt{2} - \varepsilon] \times \mathbb{R}$, with $\varepsilon > 0$, $M(t, x, y)$ is overvalued by:

$$M(\varepsilon) = \begin{bmatrix} -\sqrt{2} - \varepsilon & 1 \\ 2 & -1 - \sqrt{2} - \varepsilon \end{bmatrix}. \quad (10)$$

$M(\varepsilon) + N = \begin{bmatrix} -\sqrt{2} - \varepsilon & 1 \\ 2 & -\sqrt{2} - \varepsilon \end{bmatrix}$ is the opposite of an M-matrix, so system (8) is locally asymptotically stable and admits the following positively invariant sets

$$\mathfrak{S}(\varepsilon) = \{\varphi \in C : V(\varphi(s)) < [f_1(\varepsilon), f_2(\varepsilon)]^T, \forall\, s \in [-2, 0]\},$$

with $f_1(\varepsilon) = 3 - \sqrt{2} - \varepsilon$, and $f_2(\varepsilon) = 3\sqrt{2} - 2 + (3 - 2\sqrt{2})\,\varepsilon - \varepsilon^2$.

It yields $\mathfrak{S} = \{\varphi \in C: V(\varphi(s)) < [3-\sqrt{2}, 3\sqrt{2}-2]^T, \forall\, s \in [-2, 0]\}$

is an estimate of the domain of attraction of (8).

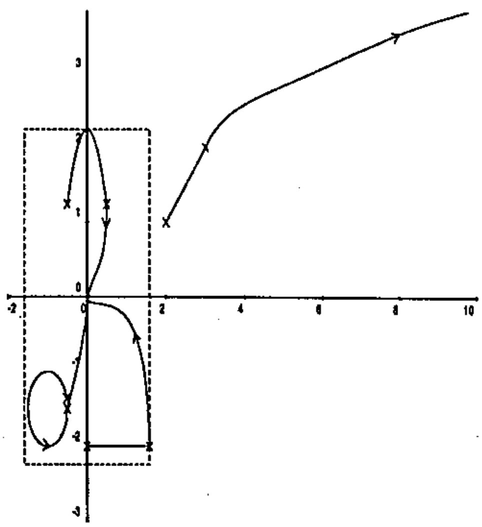

Fig 1: Simulation of syst. (8) for different initial functions

Example 2 : In order to illustrate the efficiency of Theorems 2 and 3, let us consider the system described by:

$$\dot{x}(t) = \begin{bmatrix} -2 & 2\cos^2 t \\ 1 + x_1 & -2-\sin^2 t \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ 0 & a\sin^2 t \end{bmatrix} x(t-1). \qquad (11)$$

Let us first show that the results of references [12], [13], [14], or our corollary 3, that involve constant comparison systems, are inapplicable in this case.

The finest constant comparison system associated with the vector norm $p(x) = [|x_1|, |x_2|]^T$ on any region $\mathcal{D} = \{x \in \mathbb{R}^2 : |x_1| \le \alpha\}$ {where α is any positive number) is given by:

$$\dot{z}(t) = \begin{bmatrix} -2 & 2 \\ 1 + \alpha & -1 \end{bmatrix} z(t) + \begin{bmatrix} 0 & 0 \\ 0 & |a| \end{bmatrix} z(t-1), \qquad (12)$$

and M+N is not the opposite of an M-matrix, which is needed in previous references.

However, we can apply Theorems 2 and 3 by considering the following non-constant comparison system:

$$\dot{z}(t) = \begin{bmatrix} -2 & 2 \\ 1 + \alpha & -2-\sin^2 t \end{bmatrix} z(t) + \begin{bmatrix} 0 & 0 \\ 0 & |a|\sin^2 t \end{bmatrix} z(t-1). \qquad (13)$$

There is a positive number ε (sufficiently small) such that

$$\begin{bmatrix} -2 + \varepsilon & 2 \\ 1 + \alpha & -2 + \varepsilon + (|a| - 1)\sin^2 t \end{bmatrix}$$

is the opposite of an M-matrix for any t if and only if $|a| + \alpha < 2$. So, according to Theorem 2, the solution $x = 0$ of (11) is asymptotically stable. We now apply Theorem 3 to obtain an estimate of its stability domain. On $\mathcal{D} = \{x \in \mathbb{R}^2 : |x_1| \le 2 - |a| - \varepsilon\}$, we have

$$\begin{bmatrix} -2 & 2 \\ |1 + x_1| & -2 + (|a| - 1)\sin^2 t \end{bmatrix} \le \begin{bmatrix} -2 & 2 \\ 3 - |a| - \varepsilon & -3 + |a| \end{bmatrix} = Z_2(\varepsilon),$$

and this upper bound is reached for $x_1 = 2 - |a| - \varepsilon$, and $t = 0$.

$u_c(\mathcal{D}) = \left[1, \dfrac{-1 + |a| + [(5 - |a|)^2 - 8\varepsilon]^{1/2}}{4}\right]^T$ is an importance eigenvector of $Z_2(\varepsilon)$, and $\lambda = 2 - |a| - \varepsilon$ is the biggest positive number such that the set $I_{u_c}(\lambda)$ belongs to $\mathcal{D}$. Considering ε infinitely small, we show that the set $\mathfrak{S}$ defined by:

$$\mathfrak{S} = \{\varphi \in C : p(\varphi(s)) < [2 - |a|, 2 - |a|]^T, \forall s \in [-1, 0]\}$$

is a positively invariant set that estimates the domain of stability of solution $x = 0$ of (11).

5. CONCLUSIONS

This paper has presented several original criteria to test the delay-independent stability of nonlinear, time-varying systems with delays. These results extend Borne and Gentina's work [1] to the delay case. An example shows that these criteria have less restrictive hypotheses than previous ones ([2], [12], [13], [14]). Moreover, they give (Theorem 3 and Corollary 3 in particular) estimates of the stability domain, which also represents an original contribution. Checking the algebraic conditions is rather easy, since they involve properties classically used for linear systems even if the comparison system is nonlinear. The analysis of the stability of nonlinear large-scale systems with delays does not pose more difficult problems since vector norm method is an aggregation technique (see [6]). In addition, following [13], Wang's method for robust stability can be extended to the two classes of nonlinear, time-delay systems defined in Theorems 2 and 3.

APPENDIX

Definition and properties of (–M)-matrices :
M is the opposite of an M-matrix if it is a Hurwitz matrix with non-negative off-diagonal elements. If M is the opposite of an M-matrix then ([5]):
i) M^{-1} is a non-positive matrix.
ii) M admits an eigenvector u called the *importance vector* of M, whose components are non-negative, and which is related to the real, maximal and negative eigenvalue $\lambda_m(M)$. If in addition A is irreducible then the components of u are strictly positive.
iii) for any vector $x \geq 0$, $x \neq 0$, there is an index i such that $x_i y_i < 0$ (with $y = Mx$).
A matrix M with positive off-diagonal elements is the opposite of an M-matrix if and only if the *Kotelyansky conditions* are satisfied, i.e. its successive principal minors are sign-alternate:

$$(-1)^i M\begin{bmatrix} 1, & \dots, & i \\ 1, & \dots, & i \end{bmatrix} (.) > 0 \; ; \; \forall\, i = 1, \dots, k.$$

Lemma 2 : Let $M(t, x, y)$ be the opposite of an M-matrix with all non-constant elements located on a same row. Let $\lambda_{max} = \sup[\lambda_m(M(t, x, y))\; ; (t, x, y) \in \mathcal{T}_0 \times \mathcal{D} \times \mathcal{D}\}$, and let u be an eigenvector of $M(t, x, y)$ associated with λ_{max}.
Then, the inequality :

$$M(t, x, y)\, u \leq \lambda_{max}\, u \tag{14}$$

holds for any (t, x, y) in $\mathcal{T}_0 \times \mathcal{D} \times \mathcal{D}$.

Proof : We suppose that all non-constant elements of M(.) are isolated in the last row. There are two possible cases : either (t, x, y) is such that u is an eigenvector of M(t, x, y) associated with λ_{max}, and then inequality (14) is obvious, or (t, x, y) is such that $(M(t, x, y) - \lambda_{max} I_k)$ is the opposite of an M-matrix. But the first k–1 components of vector $(M(t, x, y) - \lambda_{max} I_k).u$ are zero (due to the special form of M(.) and to the definition of u), so applying property iii), it yields the fact that its k^{th} component is strictly negative, and then inequality (14) holds again.

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