Robustness of sliding mode control under delays effects: a case study.

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ABSTRACT

Sliding mode control is a possible way to tackle the challenging problems of control. The robustness of such control with respect to nonlinearities has been demonstrated, but in presence of delays is it still a "good control"? Thus, the paper aims at studying the effects of time delays on the performances of systems controlled using sliding mode approach. Some papers are concerned with such studies [15][16][17] but only oscillations and their frequencies have been investigated for first and second order systems. Here, for higher order systems, we give some preliminary results showing that either oscillations may occur with an estimate of theirs maximal amplitudes (proportionnal to the delay and to the gain) or instability may occur for large delays or gains. For the investigated example, simulations confirm those results and suggest that:

1. some bifurcations occur: the number of switching frequencies is growing,
2. a non linear gain achieves stabilization despite the time delay control.

1. Introduction

Sliding mode control has a deep historical background: one of the reasons is that many physical systems have some discontinuity in their dynamics, as for mechanical systems with Coulomb friction (see [29]) or electrical systems with ideal relays. This has led control theorists (mostly in eastern countries) to begin with the study of some relay-based control systems. This kind of research was the starting point of the variable structure system theory: the control commutates between d different values in order to force the system flow to behave as "a non smooth contracting map", which means the motions converge to the origin with some discontinuity in the time-derivatives of the state variables. In the development of sliding mode control, which is a particular case of variable structure system control ($d = 2$), many authors (as Andronov, see [3]) introduced in the switching device some nonlinear terms depending on a small parameter ε in order to obtain a *real* qualitative behavior. Then, one makes ε tend to zero in order to derive results in sliding regime viewed as an *ideal* behavior (see [1][2]).

Based on such theory, many different control schemes have been developed (see [7][10][11][21][22][23][28][29]). For example, it is well known that if a complex system can be stated into a normal form such as (see [19])

$$\begin{cases} \dfrac{dx_i}{dt} = x_{i+1}, \quad \forall i = 1 \ldots (n-1), \\ \dfrac{dx_n}{dt} = f(t,x) + g(t,x)u, \\ y = x_1, \end{cases} \tag{1}$$

or such as (see [13])

$$\begin{cases} \dfrac{dx_i}{dt} = x_{i+1}, \quad \forall i = 1 \ldots (n-1), \\ \dfrac{dx_n}{dt} = f(t,x,u,\dot{u},\ldots,u^{(\alpha)}), \\ y = x_1, \end{cases} \tag{2}$$

then a sliding mode strategy achieves stabilization because the nonlinearities are dominated (see [5][24][26][27]) for (1) and [14] for (2)).

However, time-delays are also natural components of many engineering devices (see [25] in these proceedings), for which time history is "backwarded". They are reputed to deteriorate the stability of feedback controllers. Then, in this paper, we shall investigate some properties of sliding mode control under such *relay-delay* effects. In [15][16][17], the question of the periods of induced oscillations was studied for first and second order systems. Here, conditions for the estimation of amplitude of oscillating solutions are given for more general orders. Moreover, through an example, simulations confirm the obtained preliminaries results, suggest that (for this example) some bifurcations are occurring, and illustrate that the use of a non linear gain can suppress the time-delay inconvenience leading to an effective stabilization of the system.

2. Some preliminary results

Problem formulation

In the following, we assume that in (1) the gain function $g(t,x)$ is constant and equal to g (this assumption can be overcome, as we shall see in eqn.(15)) and that $|f(t,x)| < M_f$. Thus, selecting a linear sliding hypersurface S described by the equation

$$s(x) = \sum_{i=1}^{n} a_i x_i, \quad a_n = 1, \tag{3}$$

with the a_i coefficients determined such that $a_0 + a_1 x + \ldots + x^n$ is an Hurwitz polynomial, we apply a classical sliding mode control (if $\langle \nabla s, g \rangle \neq 0$)

$$u(t) \quad = \quad u_{\text{eq}}(t,x(t)) - \frac{k}{g}\,\text{sign}(s(t)), \tag{4}$$

$$u_{\rm eq}(t,x(t)) = -\frac{1}{g}\left(\sum_{i=1}^{n-1} a_i x_{i+1}(t) + f(t,x(t))\right), (5)$$

so that $\dot{s} = -k\,\mathrm{sign}(s)$, where $k > 0$.

However, a practical question is *"what are the qualitative behavioral changes of the system (1) with control (4) under delays effects?"*. For instance, if the output sensors cannot provide instanneous informations on the state, then

$$y(t) = h(x(t-\tau)). \qquad (6)$$

We assume that τ is a *constant* delay, and that a reconstruction of $x(t-\tau)$ is available via $y(t)$: such a reconstruction is possible either *via* a numeric approximation (see [4] for systems without delays) or *via* an observer for which separation principle or finite time convergence is valid (see [6],[9] for systems without delays). So, the applied control is also delayed, and (4) becomes

$$u(t) = u_{\rm eq}(t, x(t-\tau)) - \frac{k}{g}\,\mathrm{sign}(s(t-\tau)). \qquad (7)$$

One can conjecture that motions of (1) with control (7) will lead to some oscillating solutions, which amplitude will increase with the delay, the gain k and the speed of change of the control near the sliding surface.

A preliminary result

Let us consider $V(x(t)) = \frac{1}{2}s^2(x(t))$. The function

$$\begin{aligned}\dot{s}(t) &= \sum_{i=1}^{n-1} a_i x_{i+1}(t) - \sum_{i=1}^{n-1} a_i x_{i+1}(t-\tau) \\ &\quad + f(t,x(t)) - f(t,x(t-\tau)) \\ &\quad - k\,\mathrm{sign}(s(t-\tau)), \qquad (8)\end{aligned}$$

has a countable number of discontinuities (similar proof to [15][16][17]): then $s(t) = s(t-\tau) + \int_{t-\tau}^{t} \dot{s}(w)dw$ holds. Now using (??)

$$\begin{aligned}\dot{s}(t) &= g\Delta_t^{(t-\tau)}(u_{\rm eq}) - k\,\mathrm{sign}(s(t-\tau)), \quad (9)\\ \Delta_t^{(t-\tau)}(u_{\rm eq}) &= (u_{\rm eq}(t,x(t-\tau)) - u_{\rm eq}(t,x(t))) \quad (10)\end{aligned}$$

$$\begin{aligned}\dot{V}(x(t)) &\le \left(s(t-\tau) + \int_{t-\tau}^{t}\dot{s}(w)dw\right)\times \\ &\quad (g\Delta_t^{(t-\tau)}(u_{\rm eq}) - k\,\mathrm{sign}(s(x(t-\tau)))), \quad (11)\\ &= -k\,|s(t-\tau)| + g\Delta_t^{(t-\tau)}(u_{\rm eq})s(t-\tau) \\ &\quad + k^2\int_{t-\tau}^{t}[\mathrm{sign}(s(t-\tau))\,\mathrm{sign}(s(w-\tau))]dw \\ &\quad + g^2\int_{t-\tau}^{t}\Delta_t^{(t-\tau)}(u_{\rm eq})\Delta_w^{(w-\tau)}(u_{\rm eq})dw. \quad (12)\end{aligned}$$

Using

$$\int_{t-\tau}^{t}[\mathrm{sign}(s(t-\tau))\,\mathrm{sign}(s(w-\tau))]dw \le \tau, \qquad (13)$$

and assuming that $\left|\Delta_t^{(t-\tau)}(u_{\rm eq})\right| < M\tau$ (this is the case for example if $u_{\rm eq}(t,x)$ is at least locally Lipschitz in its second argument, τ is small, and the dynamics are bounded) leads to

$$\dot{V}(x(t)) < (gM\tau - k)\sqrt{V(x(t-\tau))} + \tau(k^2 + g^2M^2). \quad (14)$$

Note that, if we assume that in (1) the gain function $g(t,x)$ is not constant, a similar inequality may be obtained, say

$$\dot{V}(x(t)) < \omega\sqrt{V(x(t-\tau))} + \beta). \qquad (15)$$

Thus, it appears that (14) is useful. It is straightforward to see that we need $gM\tau < k$ to hold (this will be assumed throughout the rest of the paper) and that $V = y + V_\infty^2$, ($V_\infty^2 = \frac{\tau^2(k^2+g^2M^2)^2}{(k-gM\tau)^2}$) leads to

$$\begin{aligned}\dot{y}(t) &< -\frac{\alpha}{2V_\infty}y(t-\tau) \\ &\quad + \frac{\alpha y^2(t-\tau)}{2V_\infty\left(V_\infty + \sqrt{V_\infty^2 + y(t)}\right)^2}, \quad (16)\\ \alpha &= k - gM\tau. \quad (17)\end{aligned}$$

Using a linearized equation leads to $\frac{\alpha\tau}{V_\infty} < \pi$ that is

$$\sqrt{\pi}(k^2 + g^2M^2) > (k - gM\tau) > 0, \qquad (18)$$

ensuring that solutions will reach

$$\mathcal{R}_\infty = \{x \in \mathbb{R}^n : s^2(x(t)) < 2V_\infty\}, \qquad (19)$$

(only for initial values sufficiently closed to this set). But, at the price of stronger conditions, one can obtain information on the set of initial conditions for which solutions tend to $\mathcal{R}_\infty$. For this, using $y(t) = y(t-\tau) + \int_{t-\tau}^{t}\dot{y}(w)dw$ (because of the countable discontinuities of $\dot{y}(t)$) and (16) leads to:

$$\begin{aligned}\dot{y}(t) &< -\frac{\alpha}{2V_\infty}y(t) + \frac{\alpha}{2V_\alpha^3}y^2(t-\tau) \\ &\quad + \frac{\alpha}{2V_\infty}\int_{t-\tau}^{t}\dot{y}(w)dw, \quad (20)\\ \int_{t-\tau}^{t}\dot{y}(w)dw &< \int_{t-\tau}^{t} -\frac{\alpha}{2V_\infty}y(w-\tau) \\ &\quad + \frac{\alpha}{2V_\alpha^3}y^2(w-\tau)dw. \quad (21)\end{aligned}$$

Using Razumikhin's theorem (see Section 5. Appendix, and [25]) we assume that $|y(t+s)| < q\,|y(t)|, \forall s < 0$ for some $q > 1$. Then,

$$\begin{aligned}|\dot{y}(t)| &< -\frac{\alpha}{4V_\infty^2}(2V_\infty - \alpha\tau q)\,|y(t)| \\ &\quad + \frac{\alpha}{4V_\infty^4}q^2(2V_\infty + \alpha\tau)y^2(t). \quad (22)\end{aligned}$$

This leads to the convergence condition $(2V_\infty - \alpha\tau) > 0$ (with $\alpha > 0$), this is

$$\sqrt{2}(k^2 + g^2M^2) > (k - gM\tau) > 0, \qquad (23)$$

and for initial conditions in the set

$$\mathcal{I} = \left\{x \in \mathbb{R}^n : |s^2(x) - 2V_\infty| < V_\infty^2\frac{2V_\infty - \alpha\tau}{2V_\infty + \alpha\tau}\right\}. \quad (24)$$

Remark 1 *Note that condition (23) in more restrictive than the previous one (18), since (23)⇒(18).*

3. A case study: results compared to simulations

Example 1 *Consider:*

$$\begin{cases}\dot{x}_1(t) = x_2(t),\\ \dot{x}_2(t) = x_1(t)x_2(t) + u(t),\end{cases} \qquad (25)$$

using $s(x) = x_2 + 2x_1$ leads to the classic control:

$$u(t) = -x_1(t)x_2(t) - 2x_2(t) - k\,\mathrm{sign}(s(t)). \qquad (26)$$

First, if we set k to 10 and suppose that a 0.1 time delay has been neglected in the control design procedure, then one can found $M \approxeq 30$, for the given initial conditions, $V_\infty = \frac{100}{7} \approxeq 14.3$. More over as condition (23): $\sqrt{2}(k^2 +$

$g^2M^2) > (k - gM\tau) > 0, 1000\sqrt{2} > 7 > 0$ is valid (so, from remark 1, (18) is valid), then the previous results ensures that solutions initiated in the set

$$\mathcal{I} = \left\{ x \in \mathbb{R}^n : \left| s^2(x) - \frac{200}{7} \right| < 193 \right\}, \qquad (27)$$

(as soon as M does not increase!) reach the set

$$\mathcal{R}_\infty = \{ x \in \mathbb{R}^n : s^2(x(t)) < 28.6 \}. \qquad (28)$$

This is confirmed by the simulations shown in Figure 1.

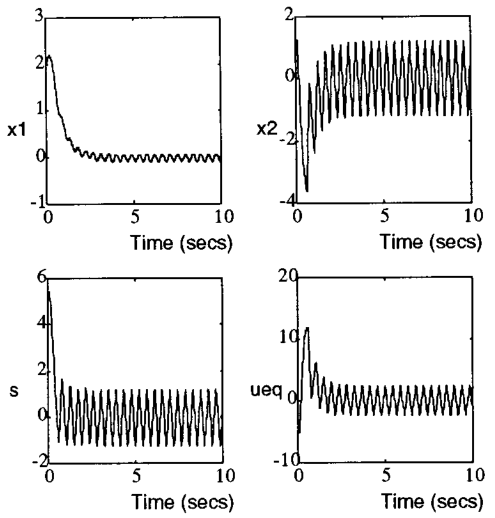


Figure 1: System (25) with control (26) $k = 10$, delayed by $\tau = 0.1$.

It is interseteing to note that x_1 and x_2 have one oscillation frequency which leads to a limit cycle (see Figure 2).

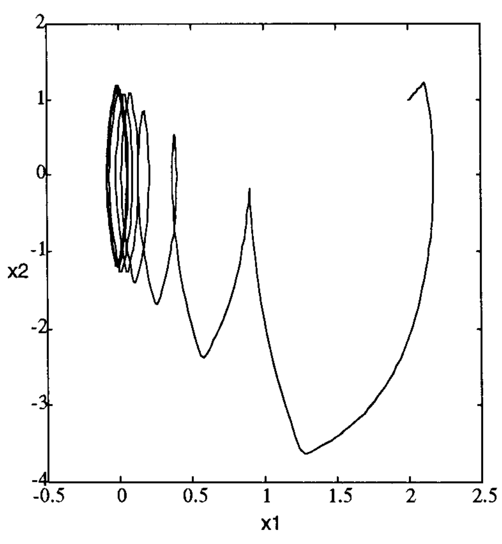


Figure 2: Phase portait of system (25) with control (26) $k = 10$, delayed by $\tau = 0.1$: convergence to a limit cycle.

Thus an interesting further investigation is the study of the limit cycle (estimation) and the determination of its frequency.

For differents values of k and τ the system converge to a band around the sliding surface (see Figure 3).

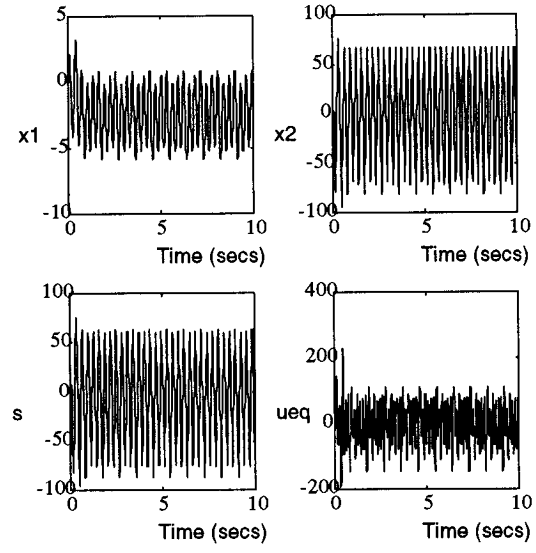


Figure 3: System (25) with control (26) $k = 1000$, delayed by $\tau = 0.08$.

But, it is important to stress that as the parametres are varying biffurcations occur: for example with $k = 1000$ and $\tau = 0.08$, x_1 and x_2 have three oscillatory frequencies (see Figures 3 and 4), leading to an asymptotic limit set with three loops (see Figure 5). Note that the parameters $k = 10$ and $\tau = 0.1$ leads to divergent motions. Lastly, from the engineering point of view, one would like to derive a sliding mode control which is less sensitive to time

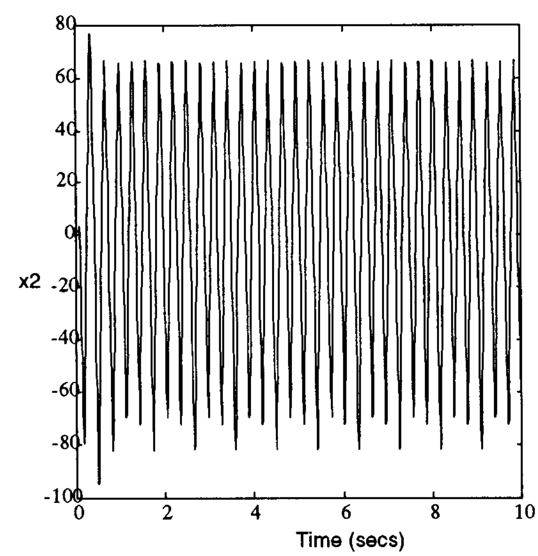


Figure 4: Zome of Figure 3 showing the period three.

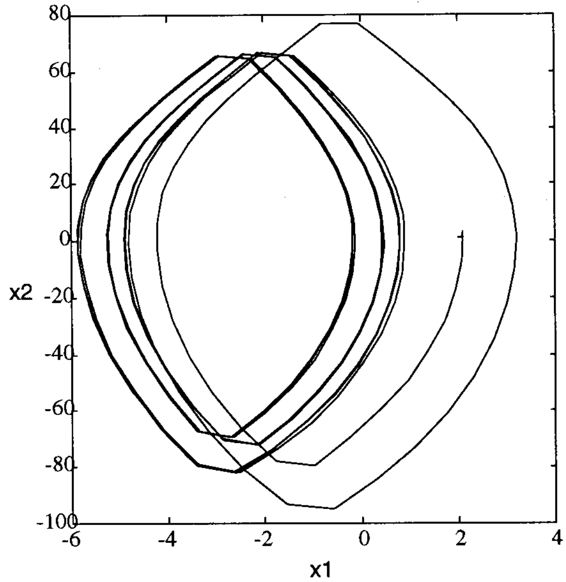
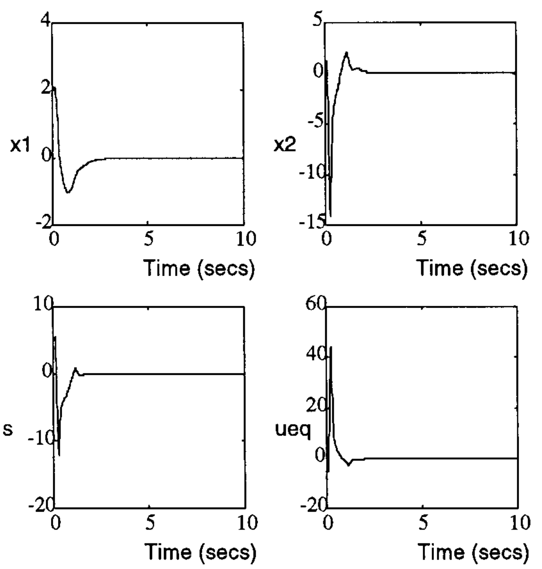


Figure 5: Phase portait of system (25) with control (26) $k = 1000$, delayed by $\tau = 0.08$: convergence to an asymptotic set with three "buckles".

delays effects. Thus on the bases of this first analysis it seems clear that if the gain k is no more linear but nonlinear one can achieve stabilization either without time delay or with small time delays.



System (25) with control (26), the following non linear gain $k(t) = 4x_1(t)(3x_1(t) + x_1^2(t))$ *and* $\tau = 0.1s$.

This is confirmed on this example : if we apply control (26) with the following non linear gain $k(t) = 4x_1(t)(3x_1(t) + x_1^2(t))$ *(the design procedure will be developp latter) and* $\tau = 0.1$*, then the applied control is*

$$u(t) = -x_1(t-\tau)x_2(t-\tau) - 2x_2(t-\tau)$$

$$-k(t-\tau)\,\mathrm{sign}(s(t-\tau)), \tag{29}$$

$$k(t-\tau) = 4x_1(t-\tau)(3x_1(t-\tau) + x_1^2(t-\tau)), \tag{30}$$

$$s(t-\tau) = x_2(t-\tau) + 2x_1(t-\tau). \tag{31}$$

and motions converge to the origin and oscillations are cancelled (see Figure 6).

4. Conclusion

These preliminary results concern the study of the sensitivity of sliding mode control with respect to time-delay effects. It is shown that, under some conditions (18) or (23), motions will reach an asymptotic limit set $\mathcal{R}_\infty$ given by (19) around the sliding surface. Moreover, we obtained some informations about the set of initial conditions guarantying the motions to converge to $\mathcal{R}_\infty$. But it is clear that, some other questions arise from this study and the obtain simulations:

1. is it possible to relax the hypothesis?
2. can we obtain an estimate of the asymptotic limit set in the phase plane (multiple "limit cycle")?
3. can we obtain some precise informations on the oscillations frequencies?
4. how can we make the sliding mode strategy insensitive to time-delays?

5. Appendix: some background on time delay-systems

A time-delay system of retarded type is a differential equation that has, in the simplest case, the form

$$\dot{x}(t) = f(t, x(t), x(t-\tau)). \tag{32}$$

This class of systems belongs to the more general class of functional differential equations. A good reference on this subject is the book by Hale's [18]. The main difference between these systems and ordinary differential equations is that there are infinite-dimensional: the state at time t is no more the vector $x(t)$ but the function denoted x_t and defined on the interval $[-\tau, 0]$ by $x_t(s) = x(t+s)$, for $s \in [-\tau, 0]$.

Concerning the stability analysis, a straightforward application of the second method of Lyapunov to time-delay systems is only possible for a restricted class of systems: in the general case, this method needs some adaptations.

The first extension of Lyapunov's method has been proposed by N. Krasovskii. In this method, the classical notion of Lyapunov function is replaced by the notion of Lyapunov-Krasovskii functional, that is a function of the state x_t and possibly the time t (see [8] or [18] for more information). As for the non delayed case, existence of a Lyapunov-Krasovskii functional is a necessary and sufficient condition for the uniform asymptotic stability.

In the second extension — Lyapunov-Razumikhin function method — the classical notion of Lyapunov function (denoted here $V(t, x(t))$) is kept but, as noticed by Razumikhin, it is worthless testing that the function V is decreasing along all the motions of the systems: it suffices to check the solutions that tends to run away from the equilibrium, *i.e.* the solutions leaving a given neighborhood of the equilibrium.

Theorem 1 (Krasovskii [20]) *Assume that* $x = 0$ *is a solution of (32),* i.e. $f(t, 0, 0) \equiv 0$. *Let* u, v, w, $p : \mathbb{R}^+ \rightarrow$

$\mathbb{R}^+$ *be continuous, non-decreasing functions, with* $u(s)$, $v(s)$, $w(s)$ *positive for* $s > 0$, *with* $u(0) = v(0) = 0$, *and* $p(s) > s$ *for* $s > 0$. *If there is a continuous function* $V : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^+$ *such that:*

1. $u(\|x\|) \leq V(t,x) \leq v(\|x\|), \forall t \in \mathbb{R}, \forall x \in \mathbb{R}^n$,
2. $\dot{V}(t,x(t)) \leq -w(\|x(t)\|)$ *if* $V(t+s, x(t+s)) < p(V(t,x(t)), \quad \forall s \in [-\tau, 0]$,

Then the null solution of (32) is uniformly asymptotically stable. ■

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