Estimate of solutions for some Volterra difference equations

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1 Introduction

Volterra difference equations (i.e. equations with discrete time [10], whose right hand side can depend on the whole previous history) arise in the mathematical modelling of some real phenomena [1][6][11][13] and also in various procedures of numerical solution of some differential and integral equations.

This motivates an essential interest in investigating the asymptotic properties of the solutions and in developing appropriate methods for the analysis. In particular, a variety of methods have been used to investigate stability of Volterra difference equations such as direct Liapunov method, comparison theorems, z-transform, etc. (see e.g. [3][4][5] and references therein). In the paper [7], topological methods were used to study stability in the first approximation of some nonlinear Volterra difference equations. The same approach will be used in this paper for estimation of the bounds on the solution. The knowledge of these bounds is quite important because they represent the error between exact solution of the original problem and its difference approximation.

To obtain these bounds we shall interpret Volterra equations as operator equations in appropriate spaces. Such approach for integral equations was used in the books [2][9][12] and for functional differential equation in the book [8].

The structure of the paper is the following. Relations defining the resolvent and the variation of constants formula for linear and nonlinear cases are given in Section 2. The estimations of the resolvent (and, in particular, summability of the resolvent without assumption of summability of the coefficients) are obtained in Section 3. Estimation of the solutions in the various spaces are given in Sections 4-6.

2 Resolvent

In this section, we consider the form of equations defining the resolvent, and also the general representation of the solution for some Volterra discrete equations.

2.1 Implicit schemes

Consider a system of linear equations

$$x_j = \sum_{k=0}^{j} a_{j,k}\, x_k + f_j, \qquad j \geq 0, \tag{1}$$

where j and k are integers, vectors $x_j \in R^n$, R^n is a linear n-dimension space equipped with some norm $|\cdot|$, $a_{j,k}$ are prescribed $n \times n$ matrices, and finally $f_j \in R^n$ are a given sequence of perturbations.

Let us assume that a unique solution x_j of system (1) does exist for all finite j. Note that a sufficient condition for the solution existence and uniqueness is the following: $det\,(I - a_{j,j}) \neq 0$ for all $j \geq 0$. Also, if the solution of system (1) does exist for arbitrary f_j, then the conditions $det\,(I - a_{j,j}) \neq 0$, $j \geq 0$ become necessary as well.

Let us find the solution x_j as a function of f_k, $k \leq j$ and auxiliary $n \times n$ matrix $r_{m,,j}, 0 \leq j \leq m$, referred to as a resolvent [9]. Multiplying both sides of equation (1) by $r_{m,,j}$ from the left and summing with respect to j from $j = 0$ to $j = m$, we obtain

$$\sum_{j=0}^{m} r_{m,j}\,(x_j - f_j) = \sum_{j=0}^{m} r_{m,j} \sum_{k=0}^{j} a_{j,k}\, x_k = \sum_{k=0}^{m} \sum_{j=k}^{m} r_{m,j}\, a_{j,k}\, x_k = \sum_{j=0}^{m} \sum_{k=j}^{m} r_{m,k}\, a_{k,j}\, x_j . \tag{2}$$

Let us require that the resolvent $r_{m,j}$ satisfies, for any m and $0 \leq j \leq m$, the relation

$$r_{m,j} = \sum_{k=j}^{m} r_{m,k}\, a_{k,j} - a_{m,j}, \quad 0 \leq j \leq m. \tag{3}$$

Then, by virtue of (2) and (3)

$$\sum_{j=0}^{m} r_{m,j}\,(x_j - f_j) = \sum_{j=0}^{m} (r_{m,j} + a_{m,j}) x_j,$$

and hence

$$\sum_{j=0}^{m} (r_{m,j} f_j + a_{m,j} x_j) = 0. \tag{4}$$

Changing the sum in the right hand side of (1) according to (4) yields the desired form of the solution

$$x_j = f_j - \sum_{k=0}^{j} r_{j,k} f_k. \tag{5}$$

So, if the solution of equations (1), (3) does exist, then it can be represented in the form (5).

Let us prove now that

$$\sum_{k=j}^{m} r_{m,k}\, a_{k,j} = \sum_{k=j}^{m} a_{m,k}\, r_{k,j}, \qquad k \le j \le m. \tag{6}$$

Substituting x_j (5) into (1), we have

$$\sum_{k=0}^{j} (r_{j,k} + a_{j,k})\, f_k = \sum_{k=0}^{j} a_{j,k} \sum_{l=0}^{k} r_{k,l}\, f_l = \sum_{l=0}^{j} \sum_{k=l}^{j} a_{j,k}\, r_{k,l}\, f_l = \sum_{k=0}^{j} \sum_{l=k}^{j} a_{j,l}\, r_{l,k}\, f_k.$$

From here and arbitrariness of f_k, since from(3) $r_{l,k}$ does not depend on the f_k, it follows that the resolvent satisfies also the equation

$$r_{m,j} = \sum_{k=j}^{m} a_{m,k}\, r_{k,j} - a_{m,j}. \tag{7}$$

Comparing (3) and (7), one verifies (6). Note that, if the matrices $a_{j,k}$ in equation (1) only depend on the difference $j - k$ (i.e., $a_{j,k} = a_{j-k}$ for all $0 \le k \le j$), then the resolvent $r_{j,k}$ also only depends on the difference $j-k$ (i.e., $r_{j,k} = r_{j-k}$) and moreover by virtue of (3) and (7),

$$r_j = -a_j + \sum_{k=0}^{j} a_{j-k}\, r_k = -a_j + \sum_{k=0}^{j} r_{j-k}\, a_k. \tag{8}$$

2.2 Nonlinear equations

Equations (3), (7) for the resolvent allow the derivation of the variation of constants formula for the nonlinear Volterra difference equation (with $G_k : R^n \to R^n$):

$$x_j = f_j + \sum_{k=0}^{j} a_{j,k} \left[x_k + G_k \left(x_k \right) \right]. \tag{9}$$

Since this equation is obtained from (1) by replacing f_j with $\left[f_j + \sum_{k=0}^{j} a_{j,k}\, G_k \left(x_k \right) \right]$, by (5) we have

$$x_j - f_j + \sum_{k=0}^{j} r_{j,k} f_k - \sum_{l=0}^{j} a_{j,l} G_l \left(x_l \right) = - \sum_{k=0}^{j} \sum_{l=0}^{k} r_{j,k} a_{k,l} G_l \left(x_l \right) = - \sum_{l=0}^{j} \sum_{k=l}^{j} r_{j,k} a_{k,l} G_l \left(x_l \right).$$

From here and (3), it follows the variation of constants formula

$$x_j = f_j - \sum_{k=0}^{j} r_{j,k} \left(f_k + G_k \left(x_k \right) \right). \tag{10}$$

2.3 Explicit schemes

In a similar way, we can obtain an expression for the solution of the Volterra equation

$$x_{j+1} = a_j\, x_j + \sum_{k=s}^{j} a_{j,k}\, x_k + f_j, \qquad j \geq s, \quad x_s = x_0. \tag{11}$$

Here, s is an initial time moment, x_0 is a prescribed initial condition, a_j is a given sequence of $n \times n$ matrices, and all other notations are the same as in equation (1). Let us denote by $R_{m,j}$ the resolvent of equation (11) which represents the $n \times n$ matrix with $R_{m,m} = I$, where I is an identity matrix.

Multiplying both sides of equation (11) by $R_{m+1,j+1}$from the left and summing in j from $j = s$ to $j = m \geq s$, we obtain

$$\sum_{j=s}^{m} R_{m+1,j+1}\, x_{j+1} = x_{m+1} - R_{m+1,s}\, x_0 + \sum_{j=s}^{m} R_{m+1,j}\, x_j$$

$$= \sum_{j=s}^{m} \left[\left(R_{m+1,j+1}\, a_j + \sum_{k=j}^{m} R_{m+1,k+1}\, a_{k,j} \right) x_j + R_{m+1,j+1}\, f_j \right].$$

Hence, the resolvent $R_{m,j}$ for any fixed m satisfies (as a function of $j \leq m$) the equation

$$R_{m+1,j} = R_{m+1,j+1} a_j + \sum_{k=j}^{m} R_{m+1,k+1}\, a_{k,j}, \; j \leq m, \; R_{m+1,m+1} = I.$$

In addition,

$$x_{m+1} = R_{m+1,s}\, x_s + \sum_{j=s}^{m} R_{m+1,j+1}\, f_j. \tag{12}$$

Let us assume that the vector $f_j \equiv 0$ in (12) and that the vector x_s has its $l-$th component equal to 1 with all other components of x_s equal to zero. Then, by virtue of (12), the $l-$th column of the resolvent satisfies equation (11) for $m \geq s$. Giving to l the values from 1 to n, we can conclude that the resolvent $R_{m,j}$ (as a function of m for any fixed j) satisfies the following homogeneous equation

$$R_{m+1,j} = a_m\, R_{m,j} + \sum_{k=j}^{m} a_{m,k}\, R_{k,j}, \; m \geq j, \; R_{j,j} = I.$$

3 Estimation of the resolvent

3.1 Convolution equation

Let us derive some estimates of the resolvent r_j for the scalar linear Volterra equation of a convolution type

$$x_j = -\sum_{k=0}^{j} a_{j-k}\, x_k, \qquad j \geq 0. \tag{13}$$

Taking into account (8), we conclude that the resolvent r_j satisfies the relation

$$r_j = a_j - \sum_{k=0}^{j} a_{j-k}\, r_k. \tag{14}$$

Theorem 1 *Assume that for all* $j \geq 0$ *and* $m \geq 0$*,*

$$a_j > 0; \quad a_{j+1} \leq a_j; \quad \frac{a_j}{a_{j+m}} \geq \frac{a_l}{a_{l+m}} \text{ for } l \geq j \geq 0. \tag{15}$$

(the last assumption in (15), meaning that the ratio a_j/a_{j+m} *for any fixed* $m > 0$ *is a nonincreasing function of* j*, is sometimes referred to as a* logarithmic convexity *of the sequence* a_j*). Let us define*

$$\alpha = \sum_{j=0}^{\infty} a_j, \quad \beta = \sum_{j=0}^{\infty} r_j. \tag{16}$$

Then

$$0 \leq r_j \leq a_j, \ \ \beta = \alpha/\left(1+\alpha\right), \ \beta/_{\alpha=\infty} = 1. \tag{17}$$

Proof. It is clear that $r_0 > 0$. Assume that $r_j \geq 0$ for $j \geq s$, but for $j = s+1$, $r_{s+1} < 0$, for the first time . Then, using (14), we have

$$0 > r_{s+1} = a_{s+1} - \sum_{k=0}^{s+1} a_{s+1-k} r_k > a_{s+1} - \sum_{k=0}^{s} a_{s+1-k} r_k = \frac{a_{s+1}}{a_s}\left[a_s - \sum_{k=0}^{s} \frac{a_{s+1-k}}{a_{s+1}} a_s r_k\right] \tag{18}$$

But, according to condition (15)

$$\frac{a_{s+1-k}}{a_{s+1}} \, a_s \leq a_{s-k}.$$

From here and (18), it follows that

$$0 > \frac{a_{s+1}}{a_s}\left[a_s - \sum_{k=0}^{s} a_{s-k} \, r_k\right] = \frac{a_{s+1}}{a_s} r_s \geq 0.$$

This contradiction shows that $r_j \geq 0$ for all $j \geq 0$. From here and (14), it follows that $r_j \leq a_j$. Hence we have proven that $0 \leq r_j \leq a_j$.

Let us turn to the proof of the second assertion (17): if $\alpha < \infty$ then, by virtue of (14),

$$\beta = \alpha - \sum_{j=0}^{\infty} \sum_{k=0}^{j} a_{j-k} \, r_k = \alpha \left(1 - \beta\right).$$

Hence $\beta = \alpha/(1+\alpha)$ for $\alpha < \infty$. Suppose now that $\alpha = \infty$. Note that for any $m > 0$ by (14), (15)

$$\sum_{j=0}^{m} r_j = \sum_{j=0}^{m} r_{m-j} \le \frac{1}{a_m} \sum_{j=0}^{m} a_j\, r_{m-j} = \frac{1}{a_m}(a_m - r_m) = 1 - \frac{r_m}{a_m} \le 1. \quad (19)$$

Consider an auxiliary function y_j defined by the equation

$$y_j = 1 - \sum_{k=0}^{j} a_{j-k}\, y_k. \quad (20)$$

By virtue of (5),

$$y_j = 1 - \sum_{k=0}^{j} r_{j-k}. \quad (21)$$

From (21) and (19), it follows that $y_j \ge 0$. Let us prove that $\beta = 1$ if $\alpha = \infty$. Assume, by contradiction, that $\beta < 1$ but $\alpha = \infty$. Then by (21) we get $y_j \ge 1 - \beta > 0$ for all $j \ge 0$. But equation (20) implies that

$$y_j \le 1 - (1-\beta) \sum_{k=0}^{j} a_{j-k} \to -\infty \quad \text{as} \quad j \to \infty.$$

This contradiction shows that $\beta = 1$ for $\alpha = \infty$. Theorem 1 is proven. ■

Remark 1 *In the case $a_{j+1} \ge a_j$, $j \ge 0$, a necessary and sufficient condition for assuring that the ratio a_j/a_{j+m} is nonincreasing with respect to j, is provided by the inequalities*

$$a_{j+2}a_j - a_{j+1}^2 \ge 0, \qquad j \ge 0,$$

because in this case, the first difference of a_j/a_{j+m} for all $j \ge 0$ is nonpositive.

3.2 Nonconvolution equation

Let us derive some estimates of the resolvent for the equation (1) in the scalar case and with the kernel $a_{j,k} = -b_k\, a_{j-k}$, where b_j and a_j are given numerical sequences. In this case by virtue of (7) the resolvent $r_{j,k}$ satisfies the equation

$$r_{j+k,k} = a_j\, b_k - \sum_{l=0}^{j} a_{j-l}\, b_{l+k}\, r_{l+k,k}. \quad (22)$$

If the solution of equation (22) is represented in the form $r_{j+k,k} = Z_{j,k}\, b_k$, then due to (22) the function $Z_{j,k}$ is defined by the equation

$$Z_{j,k} = a_j - \sum_{l=0}^{j} a_{j-l}\, b_{k+l}\, Z_{l,k}. \tag{23}$$

Let us denote

$$b = \sup_j b_j, \qquad j \geq 0.$$

Theorem 2 *Consider the scalar equation (1) with the kernel $a_{j,k} = -b_k\, a_{j-k}$. Assume that $b_j \geq 0,\ b < \infty$ and that the a_j satisfy the inequalities (15). Then,*

$$0 \leq r_{j,k} \leq b a_{j-k}, \qquad 0 \leq k \leq j. \tag{24}$$

If, in addition, either $\alpha < \infty$ or $\inf_j b_j \geq \varepsilon > 0$, then

$$\sup_k \sum_{j=0}^{\infty} r_{j+k,k} < \infty, \qquad k \geq 0. \tag{25}$$

Proof. If $b = 0$ the estimates (24), (25) are valid because in this case $r_{j,k} \equiv 0$. Because of this it is assumed below that $b > 0$. Note that for the proof of (24) it is sufficient by virtue of (22) to show that all $Z_{j,k} \geq 0$. Let us introduce the function $w_{j,k} = bZ_{j,k}$. Taking into account (23), the function $w_{j,k}$ satisfies the equation

$$w_{j,k} = ba_j - \sum_{l=0}^{j} a_{j-l}\, b_{k+l}\, w_{l,k} = f_j - \sum_{l=0}^{j} (ba_{j-l})\,(w_{l,k} + G_{l,k})\,, \tag{26}$$

with $G_{l,k} = \left(-1 + \frac{1}{b}\, b_{k+l}\right) w_{l,k},\ f_j = ba_j$.

This equation is of the type (9). Denote by y_j the resolvent of the corresponding kernel $-ba_j$. Because of (8), we have

$$y_j = b\left(a_j - \sum_{l=0}^{j} a_{j-l}\, y_l\right). \tag{27}$$

The relations (26), (27) and the variation of constants formula (10) give us

$$w_{j,k} = ba_j - \sum_{l=0}^{j} y_{j-l}\,(ba_l + G_{l,k}) = y_j - \sum_{l=0}^{j} y_{j-l} G_{l,k} = y_j + \sum_{l=0}^{j} y_{j-l}\left(1 - \frac{1}{b} b_{k+l}\right) w_{l,k}. \tag{28}$$

But $y_j \geq 0$ by virtue of Theorem 1, and $y_0 < 1$ due to (27). From here and (28) it follows that all $w_{j,k} \geq 0$. Hence, $Z_{j;k} \geq 0$. As a result, the relation (24) is proven.

For the proof of (25) it is sufficient to show that

$$\sup_k \sum_{j=0}^{\infty} w_{j,k} < \infty. \tag{29}$$

Here, $w_{j,k}$ are defined by equation (28). Let us introduce the space $\mathcal{L}$ of sequences $\left\{\varphi_j\right\}$ with a finite norm $\|\varphi\|_1$ defined as

$$\|\varphi\|_1 = \sum_{j=0}^{\infty} \left|\varphi_j\right|.$$

Define the operator L_k over the space $\mathcal{L}$ by the relation

$$L_k\varphi = \left\{\sum_{i=0}^{j} y_{j-i}\left(1 - \frac{1}{b}\, b_{k+i}\right)\varphi_i\right\},$$

where y_j is a solution of equation (27). Now, let us rewrite equation (28) in the equivalent operator form

$$\varphi^k = y + L_k\varphi^k,$$

and show that this operator equation has a unique solution. By virtue of Theorem 1 (conclusion $\beta \leq 1$) we have $\{y_j\} \in \mathcal{L}$, and also

$$\|L_k\varphi\|_1 = \sum_{j=0}^{\infty}\left|\sum_{i=0}^{j} y_{j-i}\left(1 - \frac{1}{b}\, b_{k+i}\right)\varphi_i\right| \leq \|y\|_1 \quad \|\varphi\|_1 < \infty. \tag{30}$$

Hence, the operator L_k maps $\mathcal{L}$ into itself. Let us estimate the norm $\|L_k\|$ of the operator L_k. If $\alpha < \infty$, then $\|y\|_1 < 1$ by Theorem 1. Consequently, taking into account (30), we obtain that, uniformly in k, the norm $\|L_k\| < 1$. In the case $\inf_j b_j \geq \varepsilon > 0$, we have $\sup_i \left(1 - \frac{1}{b}\, b_{k+i}\right) < 1$. Furthermore, $\|y\|_1 < 1$ by Theorem 1. Therefore, by virtue of (30), in this case $\sup_k \|L_k\| < 1$ as well. So the operator L_k is a contraction operator and $\|L_k\| < 1$ uniformly with respect to k. It means that the operator equation

$$\varphi = y + L_k\, \varphi$$

has for each k a unique solution $\varphi^k \in \mathcal{L}$ and, moreover,

$$\left\|\varphi^k\right\|_1 \leq \|y\|_1 + \|L_k\| \left\|\varphi^k\right\|_1 .$$

Hence,

$$\left\|\varphi^k\right\|_1 \leq \|y\|_1 \left[1 - \|L_k\|\right]^{-1} .$$

From here, the validity of estimate (29) follows. Theorem 2 is proven. ■

Remark 2 *Assertions (24),(25) of Theorem 2 are valid also for the kernels* $a_{j,k} = -b_{j,k}\, a_{j-k}$ *where* a_j *are the same as previously and* $0 \leq b_{j,k} \leq b_{j+1,k} \leq b < \infty$. *In addition, the inequality* $\inf_j b_j \geq \varepsilon > 0$ *is replaced by inequality* $\inf_j b_{0,j} \geq \varepsilon > 0$.

4 Estimates of the solutions

4.1 Nonlinear Volterra equations

Consider a scalar, nonlinear equation

$$x_j = f_j - \sum_{l=0}^{j} a_{j-l}\, G_l\left(x_l\right), \qquad j \geq 0. \tag{31}$$

Theorem 3 *Assume that the functions* $G_l\left(x\right)$ *are continuous with respect to* x *and that for* $i \geq j \geq m \geq 0$,

$$f_j > 0, \quad a_j > 0, \quad xG_j\left(x\right) \geq 0, \quad \frac{f_j}{f_i} \leq \frac{a_{j-m}}{a_{i-m}}. \tag{32}$$

Then the solution x_j *of (31) yields*

$$0 \leq x_j \leq f_j.$$

Note that the equivalent conditions to the last of the inequalities in (32) have a form $\frac{f_i}{f_{i+1}} \leq \frac{a_j}{a_{j+1}}$ *for all* j *verifying* $0 \leq j \leq i$.

Proof. From conditions (32) it follows that $x_0 > 0$. Assume that $x_j \geq 0$ for $0 \leq j < k$ but $x_k < 0$. Then

$$0 > x_k = f_k - \sum_{l=0}^{k} a_{k-l}\, G_l\left(x_l\right) \geq f_k - \sum_{l=0}^{k-1} a_{k-l}\, G_l\left(x_l\right) \tag{33}$$

$$= \frac{f_k}{f_{k-1}} \left[f_{k-1} - \sum_{l=0}^{k-1} a_{k-l}\, G_l\left(x_l\right) \frac{f_{k-1}}{f_k} \right].$$

But

$$a_{k-l}\, f_{k-1}/f_k \leq a_{k-1-l}.$$

Hence, due to (33), we have

$$0 > \frac{f_k}{f_{k-1}} \left[f_{k-1} - \sum_{l=0}^{k-1} a_{k-1-l}\, G_l\left(x_l\right) \right] = \frac{f_k}{f_{k-1}} x_{k-1} \geq 0.$$

This contradiction shows us that $x_j \geq 0$ for all $j \geq 0$. From here, (31) and the hypotheses of Theorem 3, it follows that $x_j \leq f_j$. Theorem 3 is proven. ■

4.2 Lipschitz condition

Consider equation (31) where all G_l are the same and equal (i.e. $G \equiv G_l,\ \ l \geq 0$).

Theorem 4 *Suppose a_j satisfy the conditions (15) of Theorem 1, the sequence f_j is bounded,*

$$\alpha = \sum_{j=0}^{\infty} a_j < \infty,$$

function $G\left(x\right)$ is such that $xG\left(x\right) \geq 0$ and satisfies Lipschitz condition with a constant C, i.e.

$$\left|G\left(x\right) - G(y)\right| \leq C\left|x - y\right|.$$

Then for all $j \geq 0$, the solution x_j of (31) verifies

$$\left|x_j - f_j\right| \leq \left\|f\right\|, \qquad \textit{with } \left\|f\right\| = \sup_j \left|f_j\right|. \tag{34}$$

Proof. Let us take arbitrary $\varepsilon > 0$ and

$$\delta = \varepsilon + \left\|f\right\|,\ \gamma\ = \sup_{|y| \leq \delta,\ k \geq 0} \left|G(y + f_k)\right|.$$

Introducing the new variable $y_j = x_j - f_j$, equation (33) takes the form

$$y_j = -\sum_{k=0}^{j} a_{j-k} G\left(y_k + f_k\right) = -\sum_{k=0}^{j} \frac{\gamma}{\varepsilon} a_{j-k}\, y_k - \sum_{k=0}^{j} \frac{\gamma}{\varepsilon} a_{j-k} \left[\frac{\varepsilon}{\gamma} G\left(y_k + f_k\right) - y_k\right]. \quad (35)$$

Let r_j be a resolvent of the kernel $-\gamma\varepsilon^{-1} a_j$ which, by virtue of (8), is defined by the equation

$$r_j = -\gamma\varepsilon^{-1}\left(a_j - \sum_{k=0}^{j} a_{j-k} r_k\right).$$

Equation (35) has the form (9) with

$$f_j = -\sum_{k=0}^{j} \frac{\gamma}{\varepsilon} a_{j-k}\, y_k, \qquad a_{j,k} = \frac{\gamma}{\varepsilon} a_{j-k}, \qquad G_k = \left(y_k + f_k\right).$$

Hence, the application of the variation of constants formula (10) gives

$$y_j = -\sum_{k=0}^{j} \frac{\gamma}{\varepsilon} a_{j-k}\ \ y_k + \sum_{k=0}^{j} r_{j-k} \frac{\gamma}{\varepsilon} \sum_{l=0}^{k} a_{k-l}\, y_l - \frac{\varepsilon}{\gamma} \sum_{k=0}^{j} r_{j-k}\, G\left(y_k + f_k\right).$$

Observe now that

$$\begin{aligned}
\sum_{k=0}^{j} r_{j-k} \sum_{l=0}^{k} a_{k-l}\, y_l &= \sum_{l=0}^{j} \sum_{k=l}^{j} r_{j-k}\, a_{k-l}\, y_l = \sum_{l=0}^{j} y_l \sum_{k=l}^{j} r_{j-k}\, a_{k-l} \\
&= \sum_{k=0}^{j} y_k \sum_{l=k}^{j} r_{j-l}\, a_{l-k} = \sum_{k=0}^{j} y_k \sum_{l=0}^{j-k} r_{j-k-l}\, a_l.
\end{aligned}$$

It means that

$$\frac{\gamma}{\varepsilon} \sum_{k=0}^{j} y_k \left[-a_{j-k} + \sum_{l=0}^{j-k} r_{j-k-l}\, a_l\right] = \sum_{k=0}^{j} r_{j-k}\ \ y_k.$$

Thus, taking into account (35) and (10), we obtain that

$$y_j = \sum_{k=0}^{j} r_{j-k} \left[y_k - \frac{\varepsilon}{\gamma} G\left(y_k + f_k\right)\right]. \quad (36)$$

Let us interpret the right-hand side of equation (36) as an operator T mapping the sequence $\varphi = \{\varphi_k\}$ into the sequence $T\varphi$ according to the formula

$$T\varphi = \sum_{k=0}^{j} r_{j-k} \left[\varphi_k - \frac{\varepsilon}{\gamma} G(\varphi_k + f_k) \right]. \tag{37}$$

Let S be a set of all φ with the norm

$$\|\varphi\| = \sup_j \left|\varphi_j\right| \leq \delta, \quad j = 0, 1, ...$$

Let us check that $TS \subset S$. Take an arbitrary $k \geq 0$ and consider the values of the difference

$$\varphi_k - \frac{\varepsilon}{\gamma} G(\varphi_k + f_k).$$

in dependence on the values $\varphi_k \in [-\delta, \delta]$. Consider three possible cases.

1) Firstly suppose that $\|f\| \leq \varphi_k \leq \|f\| + \varepsilon$. Then,

$$\varphi_k + f_k \geq 0, \quad \text{i.e. } G(\varphi_k + f_k) \geq 0,$$

and hence

$$\delta \geq \varphi_k - \frac{\varepsilon}{\gamma} G(\varphi_k + f_k) \geq \|f\| - \varepsilon,$$

because $\varepsilon\gamma^{-1} |G(\varphi_k + f_k)| \leq \varepsilon$ always holds.

2) Further, if $-\delta \leq \varphi_k \leq -\|f\|$ then $\varphi_k + f_k \leq 0$, i.e. $G(f_k + \varphi_k) \leq 0$. Consequently,

$$-\delta \leq \varphi_k \leq \varphi_k - \frac{\varepsilon}{\gamma} G(\varphi_k + f_k) \leq -\|f\| + \varepsilon.$$

3) At last if $-\|f\| \leq \varphi_k \leq \|f\|$, then

$$\left| \varphi_k - \frac{\varepsilon}{\gamma} G(\varphi_k + f_k) \right| \leq \delta. \tag{38}$$

As a result we obtain that estimate (38) is valid for any $\varphi_k \in [-\delta, \delta]$. Besides, by virtue of Theorem 1 for the resolvent r_j, relations (17) are valid for any

$\varepsilon > 0$. Therefore, using (37), (38), (17) we have $TS \subset S$ because

$$\|T\varphi\| \le \sup_j \sum_{k=0}^{j} r_{j-k} \left| \varphi_k - \frac{\varepsilon}{\gamma} G(\varphi_k + f_k) \right| \le \delta \sum_{k=0}^{\infty} r_k \le \delta.$$

Note also that the operator T is continuous on S and the closure of S coincides with S. Further, on the strength of Theorem 4 conditions and relations (17), we have

$$0 \le r_j, \quad \sum_{j=0}^{\infty} r_j < 1.$$

From here, (37) and the Lipschitz condition for G, it follows that for all sufficiently small ε, the operator T is a contractive one, i.e. equation (36) has a unique solution in S, satisfying the estimate $\|y\| \le \delta$. It means, due to the arbitrariness of ε, that inequality (34) is valid. Thus, Theorem 4 is proven. ∎

Remark 3 *In the statement of Theorem 4, it is possible to replace the hypothesis $G \equiv G_l$, $l \ge 0$ by the following, mild condition: all functions G_l are such that $G_l(x)\,x \ge 0$ and they satisfy Lipschitz conditions with the same constant C, i.e. $|G_l(x) - G_l(y)| \le C\,|x - y|, \quad l \ge 0$.*

Let C_i denote some positive constants.

Theorem 5 *Let $G_l \equiv G$ in equation (31), function $G(x)$ satisfies the conditions $xG(x) \ge 0$, $G(x) \ge -C_1$ and $a_j \ge 0, \alpha = a_1 + a_2 + ... < \infty$. Then*

$$C_2 = \|f\| + C_1\alpha \ge x_j \ge -\|f\| - \alpha G(C_2).$$

Proof. Consider the notation

$$\mathrm{sgn}x = \begin{cases} -1 \text{ for } x < 0 \\ +1 \text{ for } x > 0 \end{cases}, \qquad \mathrm{sgn}0 = \begin{cases} \mathrm{sgn}G(0) \text{ for } G(0) \ne 0 \\ +1 \text{ for } G(0) = 0 \end{cases}.$$

From equation (31), it follows that

$$x_j \le f_j - \frac{1}{2} \sum_{l=0}^{j} a_{j-l}\, G(x_l)\,(1 - \mathrm{sgn}x_l) \le \|f\| + C_1\alpha = C_2. \tag{39}$$

Similarly, by (31) and (39), we have

$$x_j \ge f_j - \frac{1}{2} \sum_{l=0}^{j} a_{j-l}\, G(x_l)\,(1 + sgn\; x_l) \ge -\|f\| - \alpha G(C_2).$$

Thus, Theorem 5 is proven. ■

Note that Theorem 5 is a consequence of the following, more general assertion.

Theorem 6 *Suppose that $G_l(x) = G(x)$ in equation (31) and the function G satisfies the conditions: $xG(x) \geq 0$ and there exist the constants $b_0 \leq 0$, $b_1 \geq 0$ such that*

$$b_1 \geq \|f\| - \alpha G(b_0), \; b_0 \geq -\|f\| - \alpha G(b_0).$$

Assume also that $a_j \geq 0 \quad \alpha = a_1 + a_2 + ... < \infty$.

Then for all $j \geq 0$, $x_j \in [b_0, \; b_1]$.

Proof. By virtue of equation (31) we have $x_0 \in [-\|f\|, \; \|f\|]$. Assume that $x_i \in [b_0, \; b_1]$ for all $0 \leq i \leq j-1$ but $x_j \notin [b_0, \; b_1]$ for the first time for $i = j$. Let us show that it is impossible. In fact by (31) similar to (39) we have

$$x_j \leq f_j - \frac{1}{2}\sum_{l=0}^{j} a_{j-l}\, G(x_l)\,(1 - sgn\; x_l) \leq \|f\| - \alpha G(b_0) \leq b_1,$$
$$x_j \geq f_j - \frac{1}{2}\sum_{l=0}^{j} a_{j-l}\, G(x_l)\,(1 + sgn\; x_l) \geq -\|f\| - \alpha G(b_1) \geq b_0.$$

Thus, Theorem 6 is proven. ■

Remark 4 *Note also that the constants b_0, b_1 in Theorem 6 do exist either if $G(x) = o(x)$ as $|x| \to \infty$, or if for two nonnegative constants P_1, P_2 such that $P_1P_2 < 1$, $G(x) = O\left(x^{P_1}\right)$ as $x \to \infty$, and $G(x) = O\left(x^{P_2}\right)$ as $x \to -\infty$.*

Theorem 7 *Suppose that, in equation (31), the perturbations $f_i \to 0$ as $i \to \infty$, the kernel satisfies $a_j > 0$, $\Delta a_j = a_{j+1} - a_j \leq 0$, $\alpha = a_1 + a_2 + ... < \infty$, and all of the functions $G_l(x)$ are equal to the same function, $G_l(x) \equiv G(x)$, where $G(x)$ is a continuous, nondecreasing function, satisfying the conditions of Theorem 5 or Theorem 6 and such that $G(x) = 0 \Leftrightarrow x = 0$.*

Then $x_j \to 0$ as $j \to \infty$.

Proof. Let us designate by M_0 and M_1, respectively, the upper and lower limits of the solution x_j, i.e.

$$M_0 = \overline{\lim}\, x_j, \qquad M_1 = \underline{\lim}\, x_j, \qquad j \to \infty.$$

By virtue of Theorem 5 or Theorem 6, both quantities M_0, M_1 are finite.

At first, let us show that $M_1 \le 0$. Assume, contrarily, that $M_1 > 0$. Then there must exist the moment $j_0 > 0$ such that $x_j > M_1/2 > 0$ for all $j \ge j_0$. From here and equation (31), it follows that for all $j > j_0$ we have

$$0 < x_j \le f_j - \sum_{l=0}^{j_0} a_{j-l}\, G(x_l) - \sum_{l=j_0+1}^{j} a_{j-l}\, G(M_1/2)$$

$$\le f_j + \max_{0 \le l \le j_0} |G(x_l)| \sum_{l=0}^{j_0} a_{j-l} \; - G(M_1/2) \sum_{l=0}^{j-j_0-1} a_l.$$

But, in the right-hand side of this inequality, the first and the second summands tend to zero as $j \to \infty$ by virtue of Theorem 7, and the last summand tends to the negative value $-\alpha G(M_1/M_2)$. This contradiction shows that $M_1 \le 0$.

Entirely similar arguments yield that $M_0 \ge 0$: in fact, if we assume that $M_0 < 0$, then there must exist a moment $j_0 > 0$ such that $x_j \le M_0/2 < 0$ for all $j \ge j_0$. Because of this for all $j > j_0$ we have

$$0 > x_j = f_j - \sum_{l=0}^{j_0} a_{j-l}\, G(x_l) - \sum_{l=j_0+1}^{j} a_{j-l}\, G(x_l)$$

$$\ge f_j + \min_{0 \le l \le j_0} G(x_l) \sum_{l=0}^{j_0} a_{j-l} \; - \sum_{l=0}^{j-j_0-1} a_l\, G(M_0/2).$$

In the right-hand side of this inequality, the first and the second summands tend to zero as $j \to \infty$, and the last summand tends to the positive value $-\alpha G(M_0/2)$. But this contradicts the left hand side of this inequality, and hence $M_0 \ge 0$.

So we have proven that $M_1 \le 0 \le M_0$. Now let us show that $M_0 = M_1 = 0$. Consider several subcases.

1) First, assume that $G(M_0) \ge -G(M_1)$. Then from the definition of M_0 there follows the existence of a sequence $j(m) \to \infty$ as $m \to \infty$ such that $x_{j(m)} \to M_0$. Further let us take an arbitrary $i > -\infty$ and consider for $i + j(m) \ge 0$ the sequence $x_{i+j(m)}$. From the boundedness of $x_{i+j(m)}$ and Bolzano-Weierstrass theorem using Cantor diagonal procedure we can construct a subsequence $j(m(k)) \to \infty$ as $k \to \infty$ such that $x_{i+j(m(k))}$ tends to some limit $\bar{x}_i$ as $k \to \infty$ for each fixed i. Besides by (31) for $i + j \ge 0$ we have

$$x_{i+j} = f_{i+j} - \sum_{l=0}^{i+j} a_{i+j-l}\, G(x_l) = f_{i+j} - \sum_{l=-j}^{i} a_{i-l}\, G(x_{l+j}).$$

Let here $j = j(m(k))$ and pass to the limit as $k \to \infty$. Then we obtain the equation for $\bar{x}_i$,

$$\bar{x}_i = -\sum_{l=-\infty}^{i} a_{i-l}\, G(\bar{x}_l), \qquad -\infty < i < \infty. \tag{40}$$

From the process of construction of $\bar{x}_i$, it follows that $\bar{x}_0 = M_0$ and $M_1 \leq \bar{x}_i \leq M_0$. If $\bar{x}_i \equiv M_0$ for all $i \leq 0$, then by virtue of (40) we have

$$\bar{x}_0 = M_0 = -\sum_{l=-\infty}^{0} a_{-l}\, G(M_0) = -G(M_0)\,\alpha.$$

From here and the assumption $\alpha > 0$, it follows that if $M_0 > 0$ then

$$0 < M_0 = -G(M_0)\,\alpha < 0.$$

This contradiction means that $M_0 = 0$. Hence because of the inequality $G(M_0) \geq -G(M_1)$, we can conclude that $G(M_1) = 0$. Therefore, $M_1 = 0$.

2) Consider, now, the case where $\bar{x}_i$ is not identically equal to M_0 for $i \leq 0$. Then since $\bar{x}_0 = M_0$, there must exist some moment $i_0 < 0$ such that $\bar{x}_{i_0} < M_0$ but $\bar{x}_{i_0+1} = M_0$. Therefore, by virtue of (40), we obtain

$$0 < \bar{x}_{i_0+1} - \bar{x}_{i_0} = -\sum_{j=-\infty}^{i_0} \left[a_{i_0+1-j} - a_{i_0-j}\right] G(\bar{x}_j) - a_0 G(\bar{x}_{i_0+1})$$
$$\leq -a_0\, G(M_0) - G(M_0) \sum_{j=-\infty}^{i_0} \left[a_{i_0+1-j} - a_{i_0-j}\right] = 0.$$

This contradiction shows that $M_0 = 0$. But $0 = G(M_0) \geq -G(M_1) \geq 0$, and hence, $M_1 = 0$.

3) Consider, at last, the case $-G(M_1) > G(M_0)$. Let us introduce a new variable $y_i = -x_i$ and a function $g(x) = -G(-x)$. Then, by virtue of (31), we obtain

$$y_i = -f_i - \sum_{j=0}^{i} a_{i-j}\, g(y_j). \tag{41}$$

In addition, the function $g(x)$ possesses exactly the same properties as $G(x)$ (i.e. $g(x)$ is continuous, nondecreasing, etc). Moreover

$$\overline{\lim}\, y_j = -M_1, \quad \underline{\lim}\, y_j = M_0, \quad -g(-M_0) = G(M_0) < -G(M_1) = g(-M_1).$$

From here and (41), it follows that with respect to the process y_i we have the case $G(M_0) \geq -G(M_1)$ that has been considered earlier.

Thus for all cases $M_0 = M_1 = 0$. Theorem 7 is proven. ■

4.3 Estimates in $\mathcal{L}_2$ of the solutions of multidimensional Volterra difference equations

Let us denote by $\mathcal{L}_2$ the Hilbert space with elements $x = (x_0, x_1, ...)$ where $x_j \in R^n$ and R^n is the Euclidean n-dimensional space with a norm $|.|$.

Consider the equation (with perturbation f)

$$x_j = f_j + \sum_{l=0}^{j} a_{j-l} G_l(x_l), \; j \geq 0, \;\; G_l : R^n \to R^n. \tag{42}$$

Here, $f = \{f_0, f_1, ...\} \in \mathcal{L}^2$, $f_j \in R^n$, i.e.

$$\|f\|^2_{\mathcal{L}_2} = \sum_{j=0}^{\infty} |f_j|^2 < \infty.$$

Assume that the functions $G_l(x)$ are continuous with respect to x and there is a constant $\gamma > 0$ such that for all $x, y \in R^n$,

$$G_l(0) = 0, \qquad |G_l(x) - G_l(y) - x + y| \leq \gamma |x - y|.$$

At last, $a = \{a_j\}$ is a given sequence of $n \times n$ matrices belonging to the space $\mathcal{L}$, *i.e.*

$$\|a\|_{\mathcal{L}} = \sum_{j=0}^{\infty} |a_j| < \infty,$$

where $|a_j|$ is a matrix norm induced by the Euclidean norm in R^n.

Let $\bar{a}(z)$ be the Laplace transformation of the kernel a of equation (42), i.e.

$$\bar{a}(z) = \sum_{l=0}^{\infty} e^{-zl} a_l,$$

where z is a complex variable.

Theorem 8 *Denote by r_j the resolvent of the kernel a. By virtue of (8), it satisfies the equation*

$$r_j = -a_j + \sum_{k=0}^{\infty} a_{j-k}\, r_k. \tag{43}$$

Let $\rho(s)$ be the spectral norm of the Laplace transformation $\overline{r}(-is)$ of the resolvent r_j, where $i = \sqrt{-1}$, $-\infty < s < \infty$. Recall that $\rho(s)$ is equal to the square root of the maximal eigenvalue of the matrix $\overline{r}'(-is)\,\overline{r}(-is)$, where $\overline{r}'$ is the complex conjugate, transposed matrix with respect to the matrix r. Let λ be as follows:

$$\lambda = \sup_s \rho(s), \qquad -\infty < s < \infty.$$

Assume that $\gamma\lambda < 1$, that all the conditions formulated above with respect to the equation (42) are fulfilled and, moreover, that

$$\textit{for } \operatorname{Re} z \geq 0, \quad \det\left[I - \bar{a}(z)\right] \neq 0. \tag{44}$$

Then the solution x_j of equation (42) satisfies the estimate

$$\|x\|_{\mathcal{L}_2} = \left(\sum_{j=0}^{\infty} |x_j|^2\right)^{1/2} \leq \delta = (1+\lambda)(1-\gamma\lambda)^{-1}\|f\|_{\mathcal{L}_2}. \tag{45}$$

Proof. Using the variation of constants formula (10), equation (42) can be rewriten in the form

$$x_j = f_j - \sum_{k=0}^{j} r_{j-k} f_k - \sum_{k=0}^{j} r_{j-k}\left(G_k(x_k) - x_k\right).$$

Let us introduce the operator Q defined on the space $\mathcal{L}_2$ by the relation

$$h_j = (Qx)_j = \sum_{k=0}^{j} r_{j-k} x_k, \qquad x \in \mathcal{L}_2, \tag{46}$$

and let us check that the operator Q maps $\mathcal{L}_2$ into itself.

Because the kernel $a \in \mathcal{L}$, by virtue of discrete analog of the Wiener-Paley theorem (see e.g. [5]) condition (44) is a necessary and sufficient one for the resolvent r to belong to $\mathcal{L}$. From here and (46) it follows that

$$\left|h_j\right| \leq \sum_{k=0}^{j} \left|r_{j-k} x_k\right| \leq \sum_{k=0}^{j} \left|r_{j-k}\right|^{1/2} \left|r_{j-k}\right|^{1/2} \left|x_k\right| .$$

Hence, using Cauchy inequality, we have

$$\left|h_j\right| \leq \left[\sum_{k=0}^{j} \left|r_k\right|\right]^{1/2} \left[\sum_{k=0}^{j} \left|r_{j-k}\right| \left|x_k\right|^2\right]^{1/2} .$$

Its means that

$$\left\|h\right\|_{\mathcal{L}_2}^2 \leq \left[\sum_{k=0}^{\infty} \left|r_k\right|\right] \sum_{j=0}^{\infty} \sum_{k=0}^{j} \left|r_{j-k}\right| \left|x_k\right|^2 = \left\|r\right\|_{\mathcal{L}}^2 \ \left\|x\right\|_{\mathcal{L}_2}^2 < \infty,$$

and, consequently, $Q\mathcal{L}_2 \subset \mathcal{L}_2$.

Therefore equation (42) can be considered as the following operator equation in the space $\mathcal{L}_2$ with respect to x

$$x = (I - Q) f - Q \left(G(x) - x\right), \qquad (47)$$
$$Q \left(G(x) - x\right) = \sum_{k=0}^{j} r_{j-k} \left[G_k(x_k) - x_k\right],$$

where I is identity operator.

Let us show that the operator in the righthand side of (47) maps the ball $\mathcal{S}_\delta \subset \mathcal{L}_2$ of radius δ into itself. For this, it is sufficient to check that $\|Q\| \leq \lambda$ because in this case for any $x \in \mathcal{S}_\delta$ we have

$$\left\|(I - Q) f - Q \left(G(x) - x\right)\right\|_{\mathcal{L}_2} \leq \left(1 + \|Q\|\right) \|f\|_{\mathcal{L}_2} + \|Q\| \gamma\delta \leq \delta. \qquad (48)$$

Let us estimate the norm of the operator Q defined by

$$\|Q\| = \sup_{x, \|x\|_{\mathcal{L}_2} = 1} \|Qx\|_{\mathcal{L}_2}, \qquad \|Qx\|_{\mathcal{L}_2} = \left[\sum_{k=0}^{\infty} \left|\sum_{j=0}^{k} r_{k-j} x_j\right|^2\right]^{1/2} .$$

Because $h \in \mathcal{L}_2$ there exists in $\mathcal{L}_2$ the Fourier transformation $h^*(s)$ of the sequence $\{h_j\}$, which by virtue of (46) is expressed as

$$h^* (s) = \sum_{l=0}^{\infty} e^{isl} \sum_{j=0}^{l} r_{l-j} x_j = \sum_{l=0}^{\infty} e^{isl} r_l \sum_{j=0}^{\infty} e^{isj} x_j = r^* (s) \, x^* (s) \, . \tag{49}$$

But the resolvent $r \in \mathcal{L}$. Hence, its $\mathcal{L}_2$ Fourier transformation $r^* (s)$ represents simultaneously the Laplace transformation, $\overline{r} (-is) = r^* (s)$. In order to calculate the Laplace transformation $\overline{r} (s)$, let us multiply both sides of equation (43) by $\exp(-sj)$ and sum with respect to j from $j = 0$ to $j = \infty$. As a result we obtain

$$\overline{r} (s) = - [I - \overline{a} (s)]^{-1} \overline{a} (s) \, . \tag{50}$$

The right hand side of (50) can be used to calculate $\rho(s)$. Further, note that by Parseval equality and (49),

$$\|h\|_{\mathcal{L}_2}^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} |h^* (s)|^2 \, ds = \frac{1}{2\pi} \int_{-\infty}^{\infty} |r^* (s) \ x^* (s)|^2 \, ds. \tag{51}$$

From (51), relation $\bar{r} (-is) = r^* (s)$ and Parseval equality, it follows that

$$\begin{aligned} \|h\|_{\mathcal{L}_2}^2 &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left| \bar{r} \ (-is) \ x^* (s) \right|^2 ds \\ &\le \lambda^2 \frac{1}{2\pi} \int_{-\infty}^{\infty} |x^* (s)|^2 \, ds = \lambda^2 \, \|x\|_{\mathcal{L}_2}^2 \, . \end{aligned}$$

Hence, $\|Q\| \le \lambda$. Consequently the operator in the righthand side of (47) maps the ball $\mathcal{S}_\delta$ into itself. In addition this operator is continuous and contractive. In fact, suppose we are given two elements $x, y \in \mathcal{S}_\delta$. Then, similar to (48), we conclude that

$$\|Q (G (x) - x) - Q (G (y) - y)\|_{\mathcal{L}_2} \le \|Q\| \, \gamma \, \|x - y\|_{\mathcal{L}_2} =\le \gamma\zeta \, \|x - y\|_{\mathcal{L}_2} \, .$$

Thus, by virtue of the principle of the contractive mapping, equation (42) has a unique solution satisfying the estimate (45). Theorem 8 is proven. ■

5 Conclusion

In this paper, some possibilities to use operator methods for investigating the solutions of Volterra difference equations have been demonstrated, and some estimates of the solutions have been derived.

Another powerful method is connected with the application of the direct Liapunov method: this is to be presented by the authors in a future paper.

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