Mean Square Stability of Difference Equations with a Stochastic Delay

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Abstract

The paper describes mean-square stability conditions for nonlinear delay difference equations with a stochastic delay. The first part develops a formula for the infinitesimal operator. Using this formula asymptotic mean square stability conditions are derived. A final example is provided.

Key words: delay systems, Markov process, stochastic stability

1 INTRODUCTION

Various problems of the theory of difference equations and their applications are considered in many papers and books. Sometimes, for the adequate modelling of real phenomena (*e.g.*, in engineering practice), difference equations with the shift of the argument (or with delay) are being used. Usually this delay is interpreted as a prescribed constant or as a known deterministic function of time. But because of the numerous sources of incomplete information, the assumption made above is not always valid. In this connection, various problems with non-exactly known values of delay could arise.

In order to take this uncertainty into consideration, sufficient conditions for independent-of-delay stability have been proposed, as well as delay-dependent ones ([3], [7] for difference equations). Such results consider that the delay is unknown (the only information is the assumption that delay is positive), or may take advantage of some knowledge concerning the delay bounds, so to yield less conservative conditions. However, such approach cannot handle the additional information about, for instance, the most probable value of the delay between these bounds.

One of the possible ways to model uncertainties in the knowledge of delay is to interprete it as a stochastic process. This approach is accepted in the paper. In what follows, we shall formulate the statement of the problem, derive the form of the infinitesimal operator and obtain asymptotic mean square stability conditions.

Stability of differential equations with a stochastic delay was investigated in a number of papers (see, *e.g.*, [1], [2], [4], [5], [6]). In these papers, stability conditions were derived under assumption that, for each fixed value of delay, the corresponding deterministic system were exponentially stable uniformly with respect to all possible values of delay. In other words, these stability conditions are obtained under assumption that the value of delay at each time moment is exactly known (*i.e.* it can be exactly measured). Below, the form of the infinitesimal operator and stability conditions are derived without this assumption and expressed immediately in terms of the probability characteristics of the delay.

Though the random case is quite important in applications, the stability analysis has appeared to be quite difficult without additional assumptions on the random driving sequences.

In particular, in this paper, delay is interpreted as the Markov process with a finite number of states and prescribed transitional probabilities which would appear to encompass many engineering situations. Our further investigation follows through the Liapunov stability techniques for difference equations and some familiarity with this theory is assumed.

2 STATEMENT OF THE PROBLEM

Let be given discrete scalar stochastic system described by the equation,

$$x(n+1) = f(n, x(n), ..., x(n - \eta(n+1)), \quad n \geq 0. \tag{1}$$

Function f depends on the arguments $(n, x(n), x(n-1), ..., x(n - \eta(n+1))$.

Stochastic process $\eta = \eta(n)$ is the Markov process with the possible values of delays $\eta = 1, 2, ..., r$ and transitional probabilities $P_{ij}(m, n)$. Functions $P_{ij}(m, n)$ $(i, j = 1, ..., r;\ m, n \geq 1)$ are conditional probabilities of the events $\{\eta(n)=j\}$ under condition $\{\eta(m)=i\}$. They statisfy the Kolmogorov-Chapman equations

$$P_{ij}(m,n)=\sum_{k=1}^{r} P_{ik}(m,s)\ P_{kj}(s,n), \quad \text{for all } m\leq s\leq n. \tag{2}$$

Initial conditions for equation (1) are defined for $n=0$ by the relations

$$\begin{aligned} x(0) &= \psi(0), \\ x(\theta) &= \psi(\theta);\ \theta=-r,-r+1,...,-1, \\ \eta(1) &= i, \qquad (1\leq i\leq r), \end{aligned} \tag{3}$$

where $\psi(0),\psi(\theta),i$ are some prescribed values. Denote by $\mathbb{H}$ the set of all possible values of η, *i.e.* $\mathbb{H}=\{1,2,...,r\}$ and by $x(n+\theta)$ the part of the sequence $x(n-1),x(n-2),...,...,x(n-r)$.

Equation (1) with initial conditions (3) given for arbitrary n, defines the Markov process $X=X(n)$ with discrete time and the set $\mathcal{B}$ of the states $\mathcal{B}=\{\eta(n),x(n),x(n+\theta)\}\in\mathbb{H}\times\mathbb{R}^{r+1}$.

Let $\varphi(k)$ be a sequence, $k\geq -r$, and let θ take all values $\{-r,-r+1,...,-1\}$ for each fixed n. Hence, $\varphi(n+\theta)$ is a part of $\varphi(k)$ for $k\in[n-r,n-1]$.

Now, let us derive the expression for the infinitesimal operator $\mathcal{L}$ of the process X. Consider scalar functions V, where

$$\begin{aligned} V &= V(n,\eta,x,\varphi(n+\theta)), \\ n &\geq 0;\eta=1,...,r;x\in\mathbb{R};\varphi(n+\theta)\in\mathbb{R}^r(\theta=-1,...,-r). \end{aligned}$$

Define the operator $\mathcal{L}_n^j$ over the functions $V(n,\eta,x,\varphi(n+\theta))$ by the formula

$$\begin{aligned} \mathcal{L}_n^j V &= \mathcal{L}_n^j V(n,\eta,x,\varphi(n+\theta)) \\ &\triangleq E[V(n+1,\eta(n+1),x(n+1),\varphi(n+1+\theta)) \\ &\quad -V(n,\eta(n),x(n),\varphi(n+\theta))]\diagup_{\eta(n)=j,\ x(n)=\varphi(n),\ x(n+\theta)=\varphi(n+\theta)}. \end{aligned} \tag{4}$$

Here $E[\xi]\diagup_A$ denotes the conditional expectation of the stochastic value ξ under condition A, and $x(n+1)$ is defined by equation (1). Because of $\eta(n+1)\in\mathbb{H}$ for any n, from (4) using (1) we obtain

$$\begin{aligned} &\mathcal{L}_n^j V(n,\eta(n),x,\varphi(n+\theta)) \\ &= \sum_{k=1}^{r} P_{jk}(n,n+1)[V(n+1,k,f(n,\varphi(n),...\varphi(n-k)),\varphi(n+1+\theta) \\ &\quad -V(n,j,\varphi(n),\varphi(n+\theta))]. \end{aligned} \tag{5}$$

Further, let us define the operator $\mathcal{L}_n V$

$$\mathcal{L}_n V = \mathcal{L}_n V(n, \eta(n), x, \varphi(n+\theta)) \triangleq \sum_{j=1}^{r} P_{ij}(0,n) \mathcal{L}_n^j V(n, \eta, x, \varphi(n+\theta)). \quad (6)$$

By virtue of (5), we have

$$\begin{aligned} & \mathcal{L}_n V(n, \eta(n), x, \varphi(n+\theta)) \quad (7) \\ = & [\sum_{k=1}^{r} V(n+1, k, f(n, \varphi(n), ..., \varphi(n-k)), \varphi(n+1+\theta))) \\ & \sum_{j=1}^{r} P_{ij}(0,n) P_{jk}(n, n+1)] \\ & - \sum_{j=1}^{r} P_{ij}(0,n) V(n, j, \varphi(n), \varphi(n+\theta)) \sum_{k=1}^{r} P_{jk}(n, n+1). \end{aligned}$$

But the last sum in (7) equals 1 for arbitrary j and n. Thus, taking into account (2), we get

$$\begin{aligned} & \mathcal{L}_n V(n, \eta(n), x, \varphi(n+\theta)) \quad (8) \\ = & \sum_{k=1}^{r} P_{ik}(0, n+1) V(n+1, k, f(n, \varphi(n), ..., \varphi(n-k)), \varphi(n+1+\theta)) \\ & - \sum_{k=1}^{r} P_{ik}(0,n) V(n, k, \varphi(n), \varphi(n+\theta)) \\ = & \sum_{k=1}^{r} \Delta W_i(n, k, \varphi(n), \varphi(n+\theta)). \end{aligned}$$

Here,

$$W_i(n, k, \varphi(n), \varphi(n+\theta)) = P_{ik}(0,n) V(n, k, \varphi(n), \varphi(n+\theta)),$$

and $\Delta W_i(n, k, \varphi(n), \varphi(n+\theta))$ is an increment of $W_i(n, k, \varphi(n), \varphi(n+\theta))$ corresponding to the change of system (1) at the n-th step under condition that the delay is fixed and equal to k.

From the relations (4) – (8) it follows the validity of the following lemma.

Lemma 1. For any $0 \leq n_1 < n_2; i = 1, ..., r; \psi(0), \psi(\theta) \in \mathbb{R}^r$, formula (9) holds:

$$E_{n_1, X_1} V(n_2, \eta(n_2), x(n_2), x(n_2+\theta)) - V(n_1, i, \psi(0), \psi(\theta)) \quad (9)$$

$$= E_{n_1,X_1} \sum_{k=n_1}^{n_2-1} \mathcal{L}_k^{\eta(k)} V(k, \eta(k), x(k), x(k+\theta)).$$

Here E_{n_1,X_1} is a sign of conditional expectation under condition

$$X_1 = X(n_1) = \{\eta(n_1) = i, \psi(0), \psi(\theta)\} \in \mathcal{B}.$$

Proof. At first, it is easily to see that

$$\begin{aligned} &V(n_2, \eta(n_2), x(n_2), x(n_2+\theta)) - V(n_1, \eta(n_1), x(n_1), x(n_1+\theta)) \\ &= \sum_{k=n_1}^{n_2-1} [V(k+1, \eta(k+1), x(k+1), x(k+1+\theta)) - \\ &-V(k, \eta(k), x(k), x(k+\theta))]. \end{aligned} \tag{10}$$

Taking expectation from both parts of (10), we obtain

$$\begin{aligned} &E_{n_1,X_1} V(n_2, \eta(n_2), x(n_2), x(n_2+\theta)) - V(n_1, i, \psi(0), \psi(\theta)) \\ &= E_{n_1,X_1} \sum_{k=n_1}^{n_2-1} [V(k+1, \eta(k+1), x(k+1), x(k+1+\theta)) \\ &-V(k, \eta(k), x(k), x(k+\theta))]. \end{aligned} \tag{11}$$

Using properties of conditional expectation, right-hand side of (11) can be represented as

$$\begin{aligned} &E_{n_1,X_1} \{ \sum_{k=n_1}^{n_2-1} E[V(k+1, \eta(k+1), x(k+1), x(k+1+\theta)) \\ &-V(k, \eta(k), x(k), x(k+\theta))] \big/ _{\eta(k),\ x(k),\ x(k+\theta)} \}. \end{aligned} \tag{12}$$

Remark that in (11)-(12), instead of the sequence $\varphi(n)$ introduced in relations (4)-(8), we have used the solution of the problem (1),(3). By virtue of this from (11),(12),(4), relation (9) holds and Lemma 1 is proven.

As for (5)-(6), right hand side of (9) can be written as follows:

$$E_{n_1,X_1} \sum_{k=n_1}^{n_2-1} \sum_{j=1}^{r} P_{ij}(n_1, k) \mathcal{L}_k^j V(k, \eta, x(k), x(k+\theta)). \tag{13}$$

Put $n_1 = 0$ in (9) and (13). Then, by virtue of (8), for any $n > 0$, $X = \{i, \psi(0), \psi(\theta)\}$,

$$E_{0,X}V(n,\eta(n),x(n),x(n+\theta))-V(0,i,\psi(0),\psi(\theta)) \tag{14}$$
$$=E_{0,X}\sum_{k=0}^{n-1}\sum_{j=1}^{r}\Delta W_i(k,j,x(k),x(k+\theta)),$$

with

$$\Delta W_i(k,j,\varphi(k),\varphi(k+\theta)) \tag{15}$$
$$=P_{ij}(0,k+1)V(k+1,j,f(k,\varphi(k),...,\varphi(k-j)),\varphi(k+1+\theta))$$
$$-P_{ij}(0,k)V(k,j,\varphi(k),\varphi(k+\theta)).$$

From (14),(15) it follows that stability problem for stochastic system (1),(3) can be reduced to the stability problems for r systems with constant delays $\eta=i$, where $i=1,...,r$.

3 STABILITY CONDITIONS

Without loss of generality stability of the solutions of system (1), (3) will be investigated under usual assumption

$$f(n,0,0,...,0)=0. \tag{16}$$

Theorem 1. Assume that there exists a function $V=V(n,\eta,x,\psi(\theta))$ such that for any sequence $\varphi(k)$ $(k\geq -r), n\geq 0, \eta\in\mathbb{H}$ and some positive constants C_j the following inequalities are met (remind ΔW_i is defined by (15))

$$1)\ C_1\varphi^2(n)\leq V(n,\eta,\varphi(n),\varphi(n+\theta))\leq C_2\max_{n-r\leq k\leq n}\varphi^2(k), \tag{17}$$

$$2)\ \sum_{j=1}^{r}\Delta W_i(n,j,\varphi(n),\varphi(n+\theta))\leq -C_3\,\varphi^2(n). \tag{18}$$

Then the zero solution of the problem (1),(3) is asymptotically mean-square stable in the whole. It means that for any

$$X=\{i,\psi(0),\psi(\theta)\}\in\mathcal{B},\ \varepsilon>0,\ n>0,$$

there is $\delta>0$ such that if $|\psi(0)|+\max_{-r\leq\theta\leq -1}|\psi(\theta)|<\delta$, then

$$E_{0,X}\ x^2(n)<\varepsilon, \tag{19}$$

$$\text{and}\ \lim_{n\to\infty}E_{0,X}\ x^2(n)=0. \tag{20}$$

Proof. From (14), using (17)-(18), it follows that

$$E_{0,X}\ x^2(n) \leq \frac{C_2}{C_1} \max_{-r\leq\tau\leq 0} \psi^2(\tau), \tag{21}$$
$$\sum_{n=0}^{\infty} E_{0,X}\ x^2(n) < \infty$$

Inequalities (21) imply the validity of relations (19), (20). Theorem 1 is proven.

Let us assume further that for some constant $K_1 > 0$ and all $n \geq 0$, x_1, $x_2 \in \mathbb{R}$, $\psi_1(\theta)$, $\psi_2(\theta) \in \mathbb{R}^r$ $(\theta = -1, -2, ..., -r)$, the following inequality is fulfilled

$$\begin{aligned} &|\ f(n, x_1, \psi_1(\theta)) - f(n, x_2, \psi_2(\theta)\ |^2 \\ &\leq K_1(|x_1 - x_2|^2 + \max_{-r\leq\theta\leq -1} |\psi_1(\theta) - \psi_2(\theta)|^2). \end{aligned} \tag{22}$$

Note that, due to (16), relation (22) implies for all $n \geq 0$, $x \in \mathbb{R}$, $\psi(\theta) \in \mathbb{R}^r$, the inequality

$$|f(n, x, \psi(\theta))|^2 \leq K_1(|x|^2 + \max_{-r\leq\theta\leq -1} |\psi(\theta)|^2). \tag{23}$$

Inequality (23) has as a consequence the following Lemma 2, which can be easily proven by induction on n.

Lemma 2. For each solution $x(n)$ of the problem (1), (3) and all n we have

$$\max_{n-r\leq s\leq n} |x(s)|^2 \leq 2^{n-1} K_2^n \left[\max_{-r\leq\tau\leq 0} |\psi(\tau)|\right]^2 \tag{24}$$
with $K_2 = \max(1, 2K_1)$.

Now, let us consider the system which can be obtained from (1), (3) for some prescribed value of delay, say $\eta = k$, where k is one of the integers $1, 2, ..., r$, *i.e.*

$$\begin{aligned} &x(n+1) = f(n, x(n), ...x(n-k)); \\ &\quad x(0) = \psi(0), x(-1) = \psi(-1), ..., x(-k) = \psi(-k). \end{aligned} \tag{25}$$

Assume that for the system (25) can be constructed a function V_k,

$$V_k = V_k(n, \varphi(n), ...\varphi(n-k))$$

satisfying the conditions (17), (18), *i.e.*

$$\begin{aligned} C_1\varphi^2(n) &\le V_k(n,\varphi(n),...,\varphi(n-k)) \le C_2\left\|\varphi_n\right\|_k^2, \qquad (26)\\ \Delta V_k &= V_k(n+1, f(n,\varphi(n),..\varphi(n-k)...,\varphi(n),...\varphi(n+1-k))\\ &\quad -V_k(n,\varphi(n),...\varphi(n-k))\\ &\le -C_3\varphi^2(n),\\ \text{with } \left\|\varphi_n\right\|_k &= \max_{n-k\le s\le n}\left|\varphi(s)\right|. \end{aligned}$$

Then the deterministic system (25) will be asymptotically stable in the whole.

Now, let us define the function V_k for the other values of delay η in such a way that stability will be kept for the stochastic case as well. Put

$$V(n,\eta,x,\psi(\theta)) \triangleq \begin{cases} V_k(n,x,\psi(-k),...,\psi(-1))/P_{kk}(0,n), \eta=k,\\ \qquad C_4x^2, \quad \eta\ne k. \end{cases} \qquad (27)$$

In addition, we assume that for all n and some positive δ,

$$P_{kk}(0,n)\ge\delta>0. \qquad (28)$$

The constant C_4 in (27) will be defined later. Let us calculate for the function (27) the expression at the left-hand side of (18). By virtue of (15) we have

$$\begin{aligned} \Delta = \sum_{j=1}^{r}[&P_{kj}(0,n+1)V(n+1,j,f(n,\varphi(n),...\varphi(n-j)),\varphi(n),...\varphi(n-j+1))\\ &-P_{kj}(0,n)V(n,j,\varphi(n),..\varphi(n-j))]. \end{aligned}$$

Because of (26), (27),

$$\Delta \le -C_3\varphi^2(n)+\Delta_1,$$

where $\Delta_1 = \sum\limits_{j,\ j\ne k}\delta_{kj}$ and, for $j\ne k$,

$$\begin{aligned} \delta_{kj} = &P_{kj}(0,n,+1)V(n+1,j,f(n,\varphi(n)),...,\varphi(n-j)),\varphi(n),...\varphi(n-j+1))\\ &-P_{kj}(0,n)V(n,j,\varphi(n),...,\varphi(n-j)). \end{aligned}$$

On the basis of (27),

$$\delta_{kj} = C_4P_{kj}(0,n+1)f^2(n,\varphi(n),...\varphi(n-j)) - C_4P_{kj}(0,n)\varphi^2(n).$$

But for all n, $\sum\limits_{j,j\neq k} P_{kj}(0,n) \leq 1-\delta$. Consequently,

$$\Delta \leq \varphi^2(n)(-C_3 + C_4K_1(1-\delta)) + C_4K_1 \sum_{j,j\neq k} P_{kj}(0,n+1)\,\|\varphi_n\|_j^2 .$$

Let us choose constant C_4 from the condition

$$-C_3 + C_4K_1(1-\delta) \leq -C_5 < 0.$$

Then, for any sequence $\varphi(n)$, the following inequality is met:

$$\Delta \leq -C_5\varphi^2(n) + K_1C_4 \sum_{j,j\neq k} P_{kj}(0,n+1)\,\|\varphi_n\|_j^2 . \tag{29}$$

Let us put $\varphi(n) = x(n)$ and substitute (29) into (14). By virtue of (26) we have for all n,

$$\begin{aligned} C_1E_{0,X}x^2(n) \leq & V(0,k,\psi(0),\psi(\theta)) \\ & +C_4K_1\sum_{s=0}^{n-1}\sum_{j,j\neq k} P_{kj}(0,s+1)E_{0,X}\,\|x_s\|_j^2 - C_5\sum_{s=0}^{n-1}E_{0,X}x^2(s). \end{aligned} \tag{30}$$

Now let us require that for all j, $j \neq k$ the series $\sum\limits_{n=0}^{\infty} P_{kj}(0,n+1)E_{0,X}\,\|x_n\|_j^2$ are convergent. Sufficient conditions for these requirement are inequalities

$$P_{kj}(0,n) \leq C^{-n}, \quad j\neq k, \quad C > 2K_2. \tag{31}$$

Then, quite similar to the proof of Theorem 1, we can prove relations (21) using inequality (30). As a result, we have proven the following result.

Theorem 2. Assume that for some $k = 1,...,r$ there exists a functional V_k,satisfying inequalities (26), and transition probability $P_{kj}(0,n)$ of the Markov chain $\eta(n)$ satisfy the inequalities (28),(31). Then the zero solution of the problem (1),(3) under condition $\eta(1) = k$ is asymptotically mean square stable in the whole.

Remark 1. Let us write the Kolmogorov-Chapman equation for the transition probabilities $P_{kj}(0,n)$ for $j \neq k$,

$$P_{kj}(0,n+1) = \sum_{s=1}^{r} P_{ks}(0,n)P_{sj}(n,n+1)$$

$$
\begin{aligned}
&= P_{kk}(0,n)P_{kj}(n,n+1) + P_{kj}(0,n)P_{jj}(n,n+1)\\
&\quad + \sum_{s,s\neq k,s\neq j} P_{ks}(0,n)P_{sj}(n,n+1).
\end{aligned}
$$

From here it follows that the functions $P_{kj}(0,n)$ for all $j, j \neq k$ tend to zero with the desired velocity if all entries $P_{sq}(n,n+1)$ of the transition probabilities matrix of the process $\eta(t)$ tend to 0 with the same velocity, except the entries of its k^{th} column which must tend to 1.

Remark 2. From the proof of Theorem 2, it follows that condition (31) can be deleted if inequalities (26) hold for $k = r$ and the last of them is replaced with

$$\Delta V_r \leq -C_3[\max_{n-r\leq s\leq n} |\varphi(s)|]^2.$$

4 EXAMPLE

Consider a scalar system

$$x(n+1) = ax\left(n - \eta(n+1)\right), \quad n \geq 0. \tag{32}$$

Here, a is a constant parameter, $|a| < 1$, and Markov process $\eta(n)$ takes two values: either 0 or 1. Probability transition matrix of the process $\eta(n)$ has a form

$$P = \begin{pmatrix} 1-c(n) & c(n) \\ d(n) & 1-d(n) \end{pmatrix}, \quad \begin{cases} c(n) = q^{-2n}, \\ d(n) = 1 - q^{-n}, \\ q > 2. \end{cases} \tag{33}$$

If $\eta(n) = 0$ and $|a| < 1$, then as a Liapunov function V, satisfying inequalities (26), we can take $V(n, \varphi(n)) = \varphi^2(n)$. Let us check now that the transition probabilities $P_{00}(0,n)$ and $P_{01}(0,n)$ satisfy the relations (28), (31) for $k = 0$ and $j = 1$.By virtue of equations (2) for $P_{00}(0,n)$ and $P_{01}(0,n)$ for $n \geq 1$, the following equalities hold:

$$
\begin{aligned}
P_{00}(0,n+1) &= P_{00}(0,n)(1-c(n)) + P_{01}(0,n)d(n),\\
P_{01}(0,n+1) &= P_{00}(0,n)c(n) + P_{01}(0,n)(1-d(n)).
\end{aligned} \tag{34}
$$

If we exclude from (34) the probability $P_{01}(0,n)$, then we obtain

$$P_{00}(0,n+1) = (1-c(n)-d(n))P_{00}(0,n) + d(n).$$

From here and (33) it follows that

$$P_{00}(0, n+1) = (q^{-n} - q^{-2n})P_{00}(0, n) + (1 - q^{-n}).$$

Hence,

$$P_{00}(0, n+1) \geq 1 - q^{-n} \geq 1 - q^{-1} > 0,$$
$$P_{01}(0, n+1) = 1 - P_{00}(0, n+1) \leq q^{-n}, \quad n \geq 1.$$

So, both relations (28) and (31) are fulfilled. It means that system (32) is asymptotically meansquare stable if $|a| < 1$ and transition probabilities of the delay $\eta(n)$ are defined by the formulas (33).

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