Estimation of the solutions of Volterra difference equations

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1. Introduction

Volterra equations, whose solution is defined by the whole previous history, are widely used in the modelling of the processes in continuous mechanics and biomechanics, problems of control and estimation and also some schemes of numerical solutions of integral and integral–differential equations [1–8].

In this connection, there is an essential interest in such properties of the solutions as stability, limiting periodicity, boundedness and various estimates of the solutions defined by the acting perturbations.

In [9], such estimates were obtained for the solutions of nonlinear scalar integral Volterra equations of convolution type

$$x(t) = f(t) + \int_0^t a(t-s)g\big(x(s)\big)\,ds, \quad t \geqslant 0.$$

Here, perturbation $f(t)$ is a function of bounded variation on $[0,\infty)$. Constructions of the paper [9] were founded on the accurate calculation of the positive and negative parts of the function $g(x)$ incoming in the above mentioned integral. Under some assumptions with respect to the kernel $a(t)$, the estimates of the solutions do not depend on the kernel $a(t)$ and are defined only by the properties of the perturbation $f(t)$.

The estimates of the paper [9] were improved in [10] for scalar linear and nonlinear Volterra integral equations.

In this paper, we consider Volterra equations with discrete time. The estimates of the solutions for both linear and nonlinear equations are derived, using some comparison theorems and auxiliary formula for representations of the solutions.

2. Nonlinear Volterra equations

Let us consider Volterra scalar nonlinear difference equation

$$x_n = \sum_{j=1}^{n} F(n, j, x_j) + f_n, \quad n \geqslant 1. \tag{1}$$

Theorem 1. *Let us assume that, in Eq.* (1), f_n *is a given sequence satisfying condition*

$$\sum_{n=1}^{\infty} |f_{n+1} - f_n| < \infty, \tag{2}$$

and $F(n, j, x)$ *is a function with the following properties:*

(P1) *continuous with respect to the third argument,*
(P2) *nonpositive for all* $x \in (-\infty, \infty)$,
(P2) *nonincreasing with respect to* n, $n \geqslant j$, *for all* $x \in (-\infty, \infty)$.

Then, the solutions x_n *of Eq.* (1) *satisfy the inequalities* (*which do not depend on* F)

$$f_n - |f_1| - \sum_{j=1}^{n-1} |f_{j+1} - f_j| \leqslant 2x_n \leqslant f_n + |f_1| + \sum_{j=1}^{n-1} |f_{j+1} - f_j|, \quad n \geqslant 1. \tag{3}$$

It is assumed that in (3) *both sums are equal to zero for* $n = 1$.

Proof. Let us introduce the function u_n defined by

$$\begin{aligned} u_n &= \sum_{j=1}^{n} F_+(n, j, x_j) + \frac{1}{2}\left(f_n - |f_1| - \sum_{j=1}^{n-1} |f_{j+1} - f_j| \right) - \epsilon n \\ &= \sum_{j=1}^{n} F_+(j, j, x_j) + \sum_{j=1}^{n} \left(F_+(n, j, x_j) - F_+(j, j, x_j)\right) \\ &\quad + \frac{1}{2}\left(f_n - |f_1| - \sum_{j=1}^{n-1} |f_{j+1} - f_j| \right) - \epsilon n. \end{aligned}$$

Here, $F_+ = \max(0, F)$ and $\epsilon > 0$ is some number. First of all, let us show that the function

$$F_+(n, j, x_j) - F_+(j, j, x_j)$$

is nonincreasing with respect to n, $n \geqslant j$, and nonpositive. Because of the inequality $xF(n, j, x) \leqslant 0$ for all $x \in (-\infty, \infty)$, we have

$$F(n, j, x) \geqslant 0 \quad \text{for } x \leqslant 0 \quad \text{and} \quad F(n, j, x) \leqslant 0 \quad \text{for } x \geqslant 0. \tag{4}$$

Further, by virtue of the conditions of Theorem 1,

$$x\big[F(n+1, j, x) - F(n, j, x)\big] \geqslant 0, \quad x \in (-\infty, \infty).$$

Therefore,

$$\begin{aligned} &F(n+1, j, x) - F(n, j, x) \leqslant 0 \quad \text{for } x \leqslant 0, \\ &F(n+1, j, x) - F(n, j, x) \geqslant 0 \quad \text{for } x \geqslant 0. \end{aligned}$$

From here and (4), it follows that

$$\begin{aligned} &F_+(l+1, j, x) - F_+(l, j, x) \leqslant 0, \quad x \leqslant 0, \\ &F_+(l+1, j, x) - F_+(l, j, x) = 0, \quad x \geqslant 0,\ l \geqslant j. \end{aligned}$$

Let us sum both parts of the last relations with respect to l from $l = j$ to $l = n - 1$. As a result, we obtain

$$\begin{aligned} &F_+(n, j, x) - F_+(j, j, x) \leqslant 0, \quad x \leqslant 0, \\ &F_+(n, j, x) - F_+(j, j, x) = 0, \quad x \geqslant 0,\ n \geqslant j. \end{aligned} \tag{5}$$

Hence, the sum

$$\sum_{j=1}^{n} \big(F_+(n, j, x_j) - F_+(j, j, x_j)\big) \tag{6}$$

is nonincreasing with respect to n and also nonpositive by virtue of (5).

Let us show further that $u_1 < 0$, by considering the different signs of f_1. From the definition of the function u_n, we get

$$u_1 = F_+(1, 1, x_1) + \frac{1}{2}\big(f_1 - |f_1|\big) - \epsilon.$$

Case 1: $f_1 > 0$. From the properties (4) of the function $F(n, j, x)$, any root of the equation

$$x_1 = F(1, 1, x_1) + f_1$$

must satisfy the condition $0 \leqslant x_1 \leqslant f_1$. Hence, $F_+(1, 1, x_1) = 0$. Consequently, $u_1 = -\epsilon < 0$ for $f_1 > 0$.

Case 2: $f_1 < 0$. Any root of the equation $x_1 = F(1, 1, x_1) + f_1$ must satisfy the condition $f_1 \leqslant x_1 \leqslant 0$. Hence, $F(1, 1, x_1) = F_+(1, 1, x_1)$ and also $-f_1 \geqslant F(1, 1, x_1) \geqslant 0$. So, in this case

$$u_1 \leqslant -f_1 + \frac{1}{2}(f_1 - |f_1|) - \epsilon = -\epsilon < 0$$

for $f_1 < 0$.

Case 3: $f_1 = 0$. If $f_1 = f_2 = \cdots = f_m = 0$, then it would be sufficient to come to the first nonzero number f_j.

Let us denote $F_- = \min(0, F)$ and introduce one more function v_n, defined by the relation

$$v_n = -\sum_{j=1}^{n} F_-(j, j, x_j) - \sum_{j=1}^{n} \big(F_-(n, j, x_j) - F_-(j, j, x_j)\big)$$
$$-\frac{1}{2}\left(f_n + |f_1| + \sum_{j=1}^{n-1} |f_{j+1} - f_j|\right) - \epsilon n, \quad n \geqslant 1,\ \epsilon > 0.$$

Let us check that $v_1 < 0$, by using the same arguments as above for the proof that $u_1 < 0$. We have

$$v_1 = -F_-(1, 1, x_1) - \frac{1}{2}(f_1 + |f_1|) - \epsilon.$$

If $f_1 > 0$, then any root of the equation $x_1 = F(1, 1, x_1) + f_1$ must satisfy the condition $0 \leqslant x_1 \leqslant f_1$. Hence, $F_-(1, 1, x_1) = F(1, 1, x_1)$ and, moreover, $-f_1 \leqslant \leqslant F_-(1, 1, x_1) \leqslant 0$, i.e., $0 \leqslant -F_-(1, 1, x_1) \leqslant f_1$. Therefore,

$$v_1 \leqslant f_1 - \frac{1}{2}(f_1 + |f_1|) - \epsilon = -\epsilon < 0.$$

If $f_1 < 0$, then any root of the equation

$$x_1 = F(1, 1, x_1) + f_1$$

will satisfy the condition $f_1 \leqslant x_1 \leqslant 0$, i.e., $F_-(1, 1, x_1) = 0$. It means that $v_1 = = -\epsilon < 0$. As a result, for all cases, $v_1 = -\epsilon < 0$. Let us show that the function

$$F_-(n, j, x_j) - F_-(j, j, x_j),$$

is a nondecreasing function of n. Since the function $xF(n, j, x)$ is nondecreasing for all $x \in (\infty, \infty)$, we can conclude that

$$x\big[F(n+1, j, x) - F(n, j, x)\big] \geqslant 0.$$

Therefore,

$$F(n+1, j, x) - F(n, j, x) \geqslant 0, \quad x \geqslant 0.$$

Besides,

$$F_-(n, j, x) = F(n, j, x), \quad x \geqslant 0.$$

It means that

$$F_-(n+1, j, x) - F_-(n, j, x) \geqslant 0.$$

From here, it follows that the function

$$F_-(n, j, x) - F_-(j, j, x), \quad n \geqslant j,$$

is a nonnegative and nondecreasing function of n, $n \geqslant j$. Hence, the sum

$$\sum_{j=1}^{n} \big[F_-(n, j, x) - F_-(j, j, x)\big] \tag{7}$$

is nondecreasing with respect to n and nonnegative. Let us remark also that, from the definitions of the functions u_n, v_n and Eq. (1), it follows that

$$x_n = \sum_{j=1}^{n} \big(F_+(n, j, x_j) + F_-(n, j, x_j)\big) + f_n = u_n - v_n. \tag{8}$$

Further, the sum (6) is nonincreasing with respect to n: then, due to (8), we get

$$\begin{aligned} u_{n+1} - u_n &\leqslant F_+(n+1, n+1, x_{n+1}) - \epsilon + \frac{1}{2}\big(f_{n+1} - f_n - |f_{n+1} - f_n|\big) \\ &\leqslant F_+(n+1, n+1, x_{n+1}) - \epsilon \\ &= F_+(n+1, n+1, u_{n+1} - v_{n+1}) - \epsilon. \end{aligned} \tag{9}$$

Similarly, because the sum (7) is nondecreasing with respect to n, we obtain for the difference $v_{n+1} - v_n$ the estimate

$$\begin{aligned} v_{n+1} - v_n &\leqslant -F_-(n+1, n+1, x_{n+1}) - \epsilon \\ &= -F_-(n+1, n+1, u_{n+1} - v_{n+1}) - \epsilon. \end{aligned} \tag{10}$$

As a result, we have obtained that $u_1 < 0$, $v_1 < 0$ and both inequalities (9), (10) are valid. Now, we have three possibilities.

Case 1: If $u_n < 0$, $v_n < 0$ for all $n \geqslant 1$ then by virtue of (8) and conditions

$$F_+(n, j, x_j) \geqslant 0, \qquad F_-(n, j, x_j) \leqslant 0,$$

we obtain

$$\begin{aligned} &\frac{1}{2}\left[f_n - |f_1| - \sum_{j=1}^{n-1} |f_{j+1} - f_j| \right] - \epsilon n \leqslant u_n \leqslant u_n - v_n = x_n \leqslant -v_n \\ &\quad \leqslant \frac{1}{2}\left[f_n + |f_1| + \sum_{j=1}^{n-1} |f_{j+1} - f_j| \right] + \epsilon n. \end{aligned}$$

From here and arbitrariness of $\epsilon > 0$, we derive the estimates (3).

Now consider two cases when inequalities $u_n < 0$ and $v_n < 0$ are not valid for all $n > 1$.

Case 2: Assume that there exists a first moment $n_0 > 1$ such that

$$u_n < 0, \quad 1 \leqslant n \leqslant n_0, \qquad u_{n_0+1} \geqslant 0, \qquad v_n \leqslant 0, \quad 1 \leqslant n \leqslant n_0,$$

but $u_{n_0+1} - v_{n_0+1} \geqslant 0$. Then we have $u_{n_0+1} - u_{n_0} \geqslant 0$. On the other hand, by virtue of (9) and (4)

$$u_{n_0+1} - u_{n_0} \leqslant F_+(n_0+1, n_0+1, u_{n_0+1} - v_{n_0+1}) - \epsilon = -\epsilon < 0.$$

Case 3: Consider now the second possible case when there exists a first moment $n_0 > 1$ such that

$$v_n < 0, \quad 1 \leqslant n \leqslant n_0, \qquad v_{n_0+1} \geqslant 0, \qquad u_n \leqslant 0, \quad 1 \leqslant n \leqslant n_0,$$

and besides $v_{n_0+1} - u_{n_0+1} \geqslant 0$. Then we have $v_{n_0+1} - v_{n_0} \geqslant 0$. On the other hand, due to (10) and (4)

$$v_{n_0+1} - v_{n_0} \leqslant -F_-(n_0+1, n_0+1, u_{n_0+1} - v_{n_0+1}) - \epsilon = -\epsilon.$$

Since cases 2 and 3 lead to contradictions, the remaining case 1 proves the estimates (3): Theorem 1 is proven. □

3. Estimates of solutions of linear Volterra equations

Let us derive the estimates of the solutions of linear scalar Volterra equation

$$x_n = \sum_{j=1}^{n} K(n, j)x_j + f_n, \quad n \geqslant 1. \tag{11}$$

Theorem 2. *Let the function $K(n, j)$ be nonpositive and nondecreasing with respect to n, $n \geqslant j$ and perturbations f_n satisfy condition* (2) *of Theorem* 1. *Then the solution x_n of Eq.* (11) *satisfies the estimates* ($f_0 = 0$)

$$\sum_{j=1}^{n} (f_j - f_{j-1})_- \leqslant x_n \leqslant \sum_{j=1}^{n} (f_j - f_{j-1})_+. \tag{12}$$

Here,

$$f_+ = \max(0, f) = \frac{1}{2}(|f| + f), \qquad f_- = \min(0, f) = f - f_+.$$

Proof. Let us represent Eq. (11) in the form

$$x_n - \sum_{j=1}^{n} K(j, j)x_j = \sum_{j=1}^{n} [K(n, j) - K(j, j)]x_j + f_n. \tag{13}$$

Let us show, using the mathematical induction method, that the solution x_n of Eq. (13) can be represented as

$$x_n = \sum_{j=1}^{n} \frac{f_j - f_{j-1}}{(1-K(n,n))\dots(1-K(j,j))} + \sum_{j=1}^{n} \frac{1}{(1-K(n,n))\dots(1-K(j,j))} \times \sum_{l=1}^{j-1} \big[K(j,l) - K(j-1,l)\big] x_l. \tag{14}$$

Here, $f_0 = 0$ and double sum at the right-hand side is equal zero for $j = 1$.

It is clear that, for $n = 1$ the value x_1 is a solution of Eq. (13). Let us assume now that formula (14) for x_n is valid for some $n > 1$ and prove its validity for $n + 1$. For this purpose let us subtract from x_{n+1} the value x_n. Using Eq. (11) we obtain

$$x_{n+1} - x_n = K(n+1, n+1)x_{n+1} + f_{n+1} - f_n + \sum_{j=1}^{n} \big[K(n+1, j) - K(n, j)\big] x_j. \tag{15}$$

Let us divide both parts of (15) by $1 - K(n+1, n+1)$. Then, taking into account the validity of the representation (14) for x_n, we must conclude that x_{n+1} can be also written in the form (14). From the representation (14) for x_n and relations $K(n+1, j) - K(n, j) \geqslant 0$, it follows that

$$(x_n)_+ \leqslant \sum_{j=1}^{n} \frac{(f_j - f_{j-1})_+}{(1-K(n,n))\dots(1-K(j,j))} + \sum_{j=1}^{n} \frac{1}{(1-K(n,n))\dots(1-K(j,j))} \times \sum_{l=1}^{j-1} \big[K(j,l) - K(j-1,l)\big] (x_l)_+. \tag{16}$$

Here, $f_0 = 0$ and double sum is equal zero for $j = 1$.

Let us take any $u_1 \geqslant (x_1)_+$ and denote by u_n the solution of the equation

$$u_n = \sum_{j=1}^{n} \frac{(f_j - f_{j-1})_+}{(1-K(n,n))\dots(1-K(j,j))} + \sum_{j=1}^{n} \frac{1}{(1-K(n,n))\dots(1-K(j,j))}$$

$$\times \sum_{l=1}^{j-1} [K(j,l) - K(j-1,l)]u_l. \tag{17}$$

This equation is of the general form $u_n = a_n + \sum_{j=1}^{n} b_j u_j$. In this form, the b_j are nonnegative since all coefficients $(1 - K(j,j))$, $[K(j,l) - K(j-1,l)]$ multiplying the u_l in the right-hand side of (17) are nonnegative. Then, we can use a comparison theorem (the proof is obvious, see also [6]) and conclude that $(x_n)_+ \leqslant u_n$. Now, let us transform equation for u_n as follows. Remark that u_{n+1}, by virtue of equality (17), can be also represented in the form

$$\begin{aligned}
&\big(1 - K(n+1,n+1)\big)u_{n+1} = (f_{n+1} - f_n)_+ \\
&\quad + \sum_{j=1}^{n} \frac{(f_j - f_{j-1})_+}{(1 - K(n,n))\dots(1 - K(j,j))} \\
&\quad + \sum_{j=1}^{n} [K(n+1,j) - K(n,j)]u_j \\
&\quad + \sum_{j=1}^{n} \frac{1}{(1 - K(n,n))\dots(1 - K(j,j))} \sum_{l=1}^{j-1} [K(j,l) - K(j-1,l)]u_l.
\end{aligned}$$

From here and relation (17) for u_n, it follows that

$$\begin{aligned}
u_{n+1} = {} & K(n+1,n+1)u_{n+1} + (f_{n+1} - f_n)_+ + u_n \\
& + \sum_{j=1}^{n} [K(n+1,j) - K(n,j)]u_j.
\end{aligned} \tag{18}$$

Let us use once more the mathematical induction method. For $n = 1$ and $n = 2$ we have by virtue of (17)

$$\begin{aligned}
u_1 &= K(1,1)u_1 + (f_1)_+, \\
u_2 &= K(2,2)u_2 + K(1,1)u_1 + (f_2 - f_1)_+ + (f_1)_+.
\end{aligned}$$

Now let us assume that the relation for u_n is valid for some $n \geqslant 2$ and show that it will be also valid for $n + 1$. According to our assumption, we have

$$u_n = \sum_{j=1}^{n} K(n,j)u_j + \sum_{j=1}^{n} (f_j - f_{j-1})_+. \tag{19}$$

If we substitute, at the right-hand side of (18), expression (19) instead of u_n, then we obtain

$$u_{n+1} = \sum_{j=1}^{n+1} K(n+1,j)u_j + \sum_{j=1}^{n+1} (f_j - f_{j-1})_+.$$

From here, it follows that function u_n for all $n \geqslant 1$ satisfies Eq. (19). But, according to the conditions of Theorem 2, the kernel $K(n, j)$ defined for $n \geqslant j \geqslant 1$, is nonpositive and also $u_n \geqslant (x_n)_+ \geqslant 0$. From here and representation (19), we conclude that

$$u_n \leqslant \sum_{j=1}^{n}(f_j - f_{j-1})_+.$$

Therefore,

$$x_n \leqslant (x_n)_+ \leqslant \sum_{j=1}^{n}(f_j - f_{j-1})_+.$$

In order to obtain the estimate of the solution x_n of Eq. (11) from below it is sufficient to introduce new variable $y_n = -x_n$. As a result we obtain

$$\sum_{j=1}^{n}(f_j - f_{j-1})_- \leqslant (x_n)_- \leqslant x_n.$$

Theorem 2 is proven. □

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