RECENT ADVANCES IN THE STUDY OF TIME DELAY SYSTEMS: MODELLING, STABILITY, STRUCTURE AND CONTROL.

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1. Introduction. Time delays are natural components of the dynamic processes in physics, mechanics, biology, ecology, physiology, economics, epidemiology, population dynamics, chemistry, aeronautics and aerospace, to name a few. Even if the process itself does not include delay phenomena, the actuators or sensors that are involved in its automatic control usually introduce such time lags.

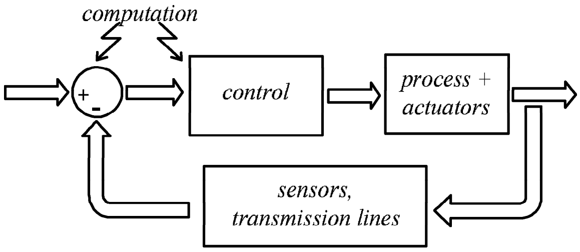

Figure 1: some sources of aftereffect.

This explains the great number of works devoted to this class of systems. During recent decades, the field of differential equations with delays, also called hereditary equations or functional differential equations (*FDEs*), has been making significant breakthroughs, and is no longer only a specialist's field: the reader can see for instance the monographs, quoted here in chronological order, by Bellman and Cooke [6] (frequency domain approach, integer functions), Krasovskii [81] (extension of the direct Lyapunov method), Halanay [59] (extension of Popov theory), Lakshmikantham and Leela [83] (differential inequalities, comparison approach), El'sgol'ts and Norkin [35] (stability, metric spaces), Driver [32], Hale [60] (topological stability methods), MacDonald [95] (biological applications), Salomon [127] (neutral-type systems, infinite-dimensional approach), Burton [15] (direct Lyapunov method, periodic solutions), Kolmanovskii and Nosov [76] (comprehensive introduction to stability, with examples), Malek-Zavarei and Jamshidi [97] (analysis, optimization), Gorecki *et al.* [55] (characteristic function, infinite-dimensional tools), Stépán [136] (characteristic function, robotics), Hino *et al.* [63] (infinite delay), Hale and Verduyn Lunel [61] (completed from [60]), Gopalsamy [48] (stability and oscillations based on ecology examples), Kolmanovskii and Myshkis [74] (deterministic and stochastic FDEs with many concrete examples), Erbe *et al.* [36] (oscillation theory), Diekmann *et al.* [31] (operator theory approach), Curtain and Zwart [20] (semi-group approach, coprime factorizations), Kolmanovskii and Shaikhet [80] (optimal control, self-adjusting systems), Gil' [45] (stability), Wang *et al.* [151] (finite-spectrum assignment), Kim [70] (I-smooth calculus) and some very recent collected works: Dugard and Verriest [33], Richard and Kolmanovskii [123], Loiseau and Rabah [92]. Of course, this list is far from complete: other contributions, in French, can be found in [108] and in some Ph.D. dissertations [129] [22][105][107][118][49][10]. One must also mention the great contribution of eastern scientists, mainly from Russia or the former Soviet Union (see

[35] or [76] and references therein). In addition to these monographs, there are nowadays many international workshops or specialized sessions, and an increasing number of papers in international journals dealing with this topic.

This interest is probably motivated by two factors. On the one hand, the fundamental aspects are quite exciting for scientists, because specific properties of delay systems are often surprising. On the other hand, the applications are of real economic interest: together with the increasing expectations of dynamic performance, engineers need the models to behave more closely to real processes, and the number of FDE models used in the sciences and in applied areas has been growing tremendously. Applications range from physiology and enzyme kinetics to whaling control and food-webs, from neural networks to laser optics, from studies of engines to the theory of business cycles, from transportation and communication systems to chemical and metallurgical processing, from traffic and power control to water resources systems, from flight mechanics to robot manipulators, from flexible structures to mechanics of viscoelasticity, from idle speed to air-fuel ratio control problems, from telerobotic systems and earth-controlled satellite devices to bio-thermo-chemical processes.

One could think that for "small" delay values, the simplest approach would consist of neglecting or replacing the delays by finite-dimensional approximations; unfortunately, ignoring effects which are adequately represented by FDEs is not a general alternative, since it can lead to potentially disastrous consequences in terms of stability and control design.

Moreover, several studies have shown that voluntary introduction of delays in feedback laws can also benefit the control (for instance, stabilization [1][50] and dead-beat control [154] of ODEs, or finite-spectrum assignment of FDEs [57][154] [10]).

The huge variety of applications gives new life to some older parts of FDE theory[1] and generates many new ones. Along with the traditional classes, new types of FDEs are being introduced and widely used in mathematical modelling, for example, stochastic FDEs, equations with impulses, hybrid and large scale FDEs, distributed (partial) FDEs, systems of non-integer dimension, equations with state dependent time lags, n-D systems, and so on. Techniques to investigate modern problems of FDE theory include many parts of real and complex analysis, functional analysis, operator theory, dynamic systems, theory of stochastic processes, theory of semi-groups, theory of systems over rings, topological methods, and more.

It would not be possible to present here all the research results and trends in such a huge field. We shall consider the reduced area of some interesting control questions related to modelling, stabilization and controllability. Lastly, some references concerning control are recalled.

2. Models for delay systems.

2.1. Functional differential equations, notion of state. A classical hypothesis in the modelling of physical processes is to assume, in the autonomous case, that the future behavior of the deterministic system can be summed up in its present *state* only. As throughout this paper, we do not consider implicit systems, this leads to Ordinary Differential Equations (ODEs), described by a n-vector $x(t)$ moving in

[1] The study of hereditary equations began during the 18th century with Bernoulli, Euler, Lagrange and Condorcet, then sporadically followed till the begining of 20th with Volterra, and others, to mention some of the most famous names. But, in the 1930's, the growing number of technical control problems showed the need of some global, mathematic statement of the question (in particular, for the initial value), which was provided by the paper of Myshkis in 1949 [106], who defined the notion and classification of FDEs.

Euclidean space R^n:

$$\text{(1)} \qquad \begin{aligned} \dot{x}(t) &= f\left(x(t), t, u(t)\right), \qquad t \geq t_0; \\ x(t_0) &= x_0 \in R^n. \end{aligned}$$

Dots indicate the time-derivatives, and $u(t)$ denotes the input (control or disturbances).

However, in numerous cases (see many examples in [74]), some "aftereffect" cannot be neglected in the modelling, which means one has to take into account an irreducible influence of the past. Iit is clear that, for instance, the simple delay equation

$$\text{(2)} \qquad \dot{x}(t) = -x(t-h),$$

has several solutions (for $h = \pi/2$: $\sin t$, $\cos t$,...) that achieve the same value at an infinite number of instants. Then, the *state* cannot anylonger be a vector $x(t)$ defined at a discrete value of time t; in Functional Differential Equations (FDEs), the state is a function x_t corresponding to the past time-interval $[t-h,\, t]$, where h is a positive, irreducible-to-zero constant (Shimanov's notation, 1960).

This *argument deviation*, *i.e.* the "time-delay" h, may be finite or infinite: it represents the maximal value of all the (possibly time-varying) delay phenomena in the process, a "memory time horizon". Note that the equations as (2) are also called *differential-difference* equations, since both kinds of operators are involved.

Two classes of hereditary models are considered: the retarded systems and the neutral ones (the third, mathematical class of *advanced* systems, $h < 0$, is not considered in this study for reasons of non-causality).

Retarded systems with input $u(t)$ can generally be described by FDEs as

$$\text{(3)} \qquad \begin{aligned} \dot{x}(t) &= f\left(x_t, t, u_t\right), \qquad t \geq t_0, \\ x_t(\theta) &= x(t+\theta), \qquad -h \leq \theta \leq 0, \\ u_t(\theta) &= u(t+\theta), \qquad -h \leq \theta \leq 0, \\ x(\theta) &= \varphi(\theta), \qquad t_0 - h \leq \theta \leq t_0. \end{aligned}$$

The nature of physics is known to be nonlinear, and such equations arise very often in the literature (see [74]). The vector $x(t)$ will be called, here, the *solution at time* t (it is also called the "instantaneous state" [76]). Note that the functional notation x_t needs the initial condition φ for equation (3) to be prescribed on the interval $[-h, 0]$. In fact, it is more natural to consider as state space the set of continuous functions mapping the interval $[-h, 0]$ into R^n, denoted $\mathcal{C}$ throughout this paper. Under certain conditions on f [60], for a given, continuous function $u(t)$, $t \in R$, and any given function y of $\mathcal{C}$, there exists a unique solution of (3) such that $x(t) = z(t)$ for $t \in [-h, 0]$ (here, z may mean an initial condition). This result can easily be obtained using a step-by-step method. For instance, let us consider (2), the initial function $\varphi(\theta) = 1$ for all $\theta \in [-h, 0]$. Then, equation (2) on the first time interval $[0, h]$ gives $\dot{x}(t) = 1$ or $x(t) = 1 - t$. Expression of $x(t)$ can then be obtained on $[h, 2h]$ by using the same scheme, and so on. The resulting solution is a succession of polynomial functions of t, in increasing degree at each interval $[kh, (k+1)h]$ (see Figure 2).

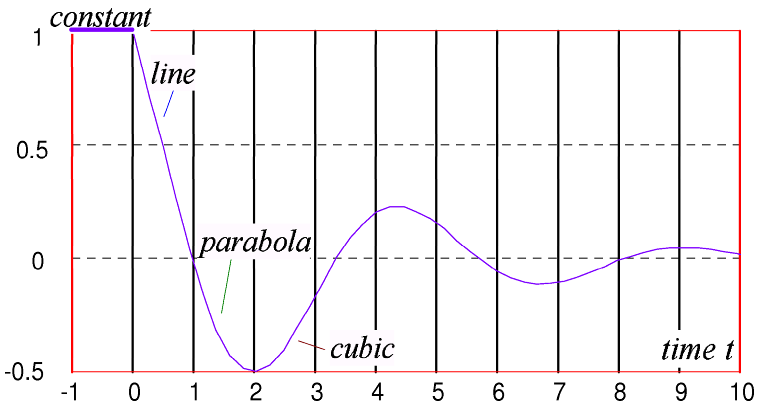

Figure 2: solutions of system (2), $h = 1$.

Classification as a *neutral* system implies that in the modelling procedure, the same highest derivative order concerns some component of $x(t)$ at both time t and past time(s) $t' < t$, which implies an increased mathematical complexity. The following system is of neutral type:

$$\dot{x}(t) = f\left(x_t, t, \dot{x}_t, u_t\right), \tag{4}$$

or in Hale's form [60]

$$F\dot{x}_t = \frac{dFx_t}{dt} = f\left(x_t, t, u_t\right), \tag{5}$$

where $F : \mathcal{C} \to R^n$ is some regular (to avoid implicit systems) operator with retarded argument on time, for instance,

$$Fx_t = x(t) - Dx(t - \omega),$$

with D a constant matrix. Such models may arise from the approximation of hyperbolic, distributed parameter equations with mixed initial and derivative boundary conditions, such as wave propagation in processes including steam pipes or zero-loss transmission lines [61, 76, 74, 105]. They can also be encountered, however, in robotics (manipulators in contact with a rigid environment [113]). In this case, due to the contained difference-equation involving $\dot{x}(t)$, the trajectory may "replicate", at any time, any time-discontinuity of the initial condition $\varphi(t)$, even if f and D present many smoothness properties (while the solutions of retarded systems will be smoothed with increasing time, that is, the differentiability degree of the solution increases with time).

Nature is nonlinear, we said, but linear models are very useful: they are a bit easier to deal with and constitute a good basis for investigating many properties of delay systems. The time-invariant model is

$$\begin{aligned} \dot{x}(t) = & \sum_{k=1}^{q} D_k\, \dot{x}(t - \omega_k) \\ & + \sum_{i=0}^{k} \left(A_i x(t - h_i) + B_i u(t - h_i)\right) \end{aligned} \tag{6}$$

$$+\sum_{j=1}^{r}\int_{t-\tau_j}^{t}(C_j(\theta)\,x(\theta)+G_j(\theta)\,u(\theta))\,d\theta,$$

where $h_0 = 0$, the matrix A_0 (constant) represents *instantaneous* feedback gains; A_i, $i > 0$ (constant), represent *discrete-delay* phenomena; the last integrals correspond to *distributed-delay* effects, whose influence is weighted by the C_j over the time intervals $[t-\tau_j,\,t]$; D_i are the *neutral* part, and B_i, $G_j(s)$ are input matrices. Here, $h = \max_{i,j,k}\,\{h_i, \tau_j, \omega_k\}$. Many physical systems can also be approximated by such models (see for instance [41] and references herein) with, mainly, only one neutral delay ($q = 1$).

Note that, in (6), $C_j \equiv -C_k$ for some (j,k) permits one to consider "discrete-and-distributed" effects such as $\int_{t-\tau_j}^{t-\tau_k} C_j x(\theta)d\theta$, and also that some additional approximation may allow the distributed effects to be replaced by a sum of discrete ones, by considering that

$$\int_{t-\tau}^{t} C(\theta)\,x(\theta)d\theta \approx \frac{\tau}{d}\sum_{i=1}^{d}\alpha_i C(\frac{i\tau}{d})x(t-\frac{i\tau}{d}),$$

with constant coefficients $\alpha_i \in R$.

Due to this simplification, many results deal with the particular case of discrete-delay systems defined by

$$\dot{x}(t) = A_0 x(t) + \sum_{i=1}^{k} A_i x(t-h_i) + \sum_{i=1}^{k} B_i u(t-h_i). \tag{7}$$

The special class of systems with *commensurate delays* is particularly investigated, where delays $h_i = i\delta$ are all integer multiples of a same positive, constant, basic delay δ (then, $h = k\delta$). In the following, we shall see that several important results are now available for the design of linear models with commensurate delays.

2.2. Solution of linear, discrete-delay systems. There exist several numerical methods (see [21]) for the construction of approximate solutions for FDEs, which mainly use step-by-step approaches, *i.e.*, iterative resolution over time intervals $[jh, (j+1)h]$ by means of classical ODEs procedures (Euler, Runge-Kutta,...), with continuity at instants jh. Concerning the general solution of (3), we know [76][61] that some Lipschitz properties on f ensure the existence of a unique solution for given φ and $u_{t\in R}$. Of course, as in the ODE case, the explicit solution is not known, but we can illustrate here the question of linear systems with commensurate delays, described by:

$$\begin{aligned}\dot{x}(t) &= \sum_{i=0}^{k}(A_i x(t-i\delta) + B_i u(t-i\delta)) \\ y(t) &= \sum_{i=0}^{k} C_i x(t-i\delta), \quad x(\theta) = \varphi(\theta)\ (-k\delta \le \theta \le 0),\end{aligned} \tag{8}$$

which solution [142][97] from $t_0 = 0$ is

$$x(t;t_0,\varphi,u) = F(t)\varphi(0) + \sum_{i=0}^{k}\int_{-i\delta}^{0} F(t\text{-}\theta\text{-}i\delta)A_i\varphi(\theta)d\theta$$

$$+\int_0^t F(t-\theta)\left[\sum_{i=0}^{k} B_i u(\theta - i\delta)\right] d\theta.$$

Here, $F(t)$ is the *fundamental matrix*, solution of $\dot{F}(t) = \sum_{i=0}^{k}(A_i F(t-i\delta))$, $F(0) = I$, $F(t<0) = 0$. Of course, the main difficulty is to calculate $F(t)$. It can be done, by means of infinite-series development, with the *Kirillova-Churakova operators*,

$$P_{q+1}(j) = \sum_{i=0}^{k} A_i P_q(j-i), \qquad (9)$$
$$P_0(0) = I,\ P_q(j) = 0 \text{ for } i \text{ or } j < 0.$$

Then, for any integer $\lambda > 0$,

$$F(t) = \sum_{j=0}^{\lambda}\sum_{q=j}^{\infty} \frac{1}{q!} P_q(j)(t-j\delta)^q \text{ for } t \in [0, \lambda\delta].$$

2.3. Operators in infinite dimension. It is possible to include delay systems within the larger class of infinite-dimensional systems. This approach, based on some abstract state space formulation in terms of operators, may benefit from the appropriate definitions of controllability/stabilizability, observability/detectability, etc. that have been defined in this very general framework. Among the extensive literature concerning these models, see [7, 20, 29, 28, 69, 99]. For simplicity, following [129], we reduce this presentation to linear, single-delay systems, say

$$\begin{aligned} \dot{x}(t) &= A_0 x(t) + A_1 x(t-h) + B_0 u(t), \\ y(t) &= C_0 x(t). \end{aligned} \qquad (10)$$

We denote by $\mathcal{L}_2 = \mathcal{L}_2([-h, 0]; R^n)$ the set of square-integrable functions $[-h, 0] \to R^n$ (the solutions of delay systems in infinite dimension can be reduced [99] to initial functions φ belonging to $\mathcal{L}_2$). The behavior is represented by the variable $\overline{x}$, $\overline{x}(t) = [x(t), x_t] = [x^0(t), x^1(t)]$, belonging to the Hilbert space $\mathcal{M}_2 = \mathcal{M}_2([-h, 0]; R^n) = R^n \times \mathcal{L}_2$. Then, system (10) can be described by

$$\begin{aligned} \dot{\overline{x}}(t) &= \widetilde{A}\overline{x}(t) + \widetilde{B}u(t), \\ y(t) &= \widetilde{C}\overline{x}(t), \end{aligned} \qquad (11)$$

where the operator $\widetilde{A}$ is unbounded, closed, and dense in the space $\mathcal{M}_2$, and is defined by

$$\overline{x}(t) \to \widetilde{A}\,\bar{x}(t) = [A_0 x^0(t) + A_1 x^1(t)(-h),\ \frac{dx^1(t)(\theta)}{d\theta}],$$

while the operators $\widetilde{B}$, $R^m \to \mathcal{M}_2$, and $\widetilde{C}$, $\mathcal{M}_2 \to R^p$, are bounded and defined by

$$\begin{aligned} u(t) &\to \widetilde{B}u(t) = [B_0 u(t),\ 0], \\ \overline{x}(t) &\to \widetilde{C}\overline{x}(t) = y(t). \end{aligned}$$

This representation allows one to use the properties of semi-groups and, from the general theory of infinite-dimensional differential equations, the solution is uniquely defined by

$$\overline{x}(t; \overline{x}_0, u) = S(t)\overline{x}_0 + \int_0^t S(t-\theta)\widetilde{B}u(\theta)d\theta \qquad (12)$$

where the family $\{S(t), t \geq 0\}$ is the continuous semigroup of operators spanned by $\widetilde{A}$ [60], satisfying

$$S(0) = I, \qquad S(t+s) = S(t)S(s),$$
$$\frac{d}{dt}\left(S(t)\overline{x}_0\right) = \widetilde{A}S(t)\overline{x}_0 \qquad t, s \geq 0.$$

This approach is then a direct extension of the exponential solutions for unbounded operators (if $h = 0$, then $\widetilde{A}$ is bounded and $S(t) = e^{\widetilde{A}t}$). We considered here a simple class of discrete delays. However, infinite dimension allows distributed delays to be considered as well, with an unmodified notation. This may be the main advantage of these general models, but it also presents some drawbacks; for instance, all control laws that are obtainable from such a formalism are designed under an undiscernible, distributed form, $u(t) = \int_{-h}^{0} F(\theta)x(t+\theta)d\theta$, that does not allow any *a priori* preference of a (often more easy to implement) discrete feedback.

2.4. Geometric approach: systems over polynomial rings. For nonlinear systems in particular, FDEs appear as a very convenient tool, with a good compromise between generality and simplicity. For linear systems (and only in this case), the classical geometric approach (in the sense of Wonham) for linear ODEs has been generalized, up to a certain point, to systems with coefficients over a ring [137]. Then, the basic idea is to translate these results into the context of delay systems [18, 19, 66, 86, 104, 143]: instead of defining vectors and matrices over the field of real numbers R, leading to the vector space R^n, one uses $\mathbf{R} = R[\nabla]$[2], the commutative ring of polynomials in the delay operator ∇ with, for commensurate-delays system (8), $\nabla(f)(t) = f(t - \delta)$. Then, (8) becomes

$$\begin{aligned} \dot{x}(t) &= \mathbf{A}(\nabla)x(t) + \mathbf{B}(\nabla)u(t), \\ y(t) &= \mathbf{C}(\nabla)x(t), \\ \mathbf{A}(\nabla) \in R^{n\times n}[\nabla], &\mathbf{B}(\nabla) \in R^{n\times m}[\nabla], \mathbf{C}(\nabla) \in R^{p\times n}[\nabla] \end{aligned} \tag{13}$$

where x belongs to the free state-module $\mathbf{R}^n = R[\nabla]^n$, u and y to the associated input and output free modules (a *module* is the analogue of a vector space over a ring). Obviously, the absence of an inverse in $R[\nabla]$ corresponds to the impossibility of realizing the advance operator ∇^{-1}. The main advantage of such modelling is its apparent finite dimension: in addition, since $R[\nabla]$ is a principal ideal domain[3], many results concerning the Smith form and invariant polynomials can be used in the realization theory. The solutions are directly determined by inversion of the Laplace transform:

$$\begin{aligned} y(s) &= C_s(sI - A_s)^{-1}B_s u(s) + C_s(sI - A_s)^{-1}\phi(s), \\ M_s &\triangleq \mathbf{M}(e^{-\delta s}) \text{ for } M = A, B, \text{or } C. \end{aligned} \tag{14}$$

[2]We shall use the classical notations $R[\nabla]$ for the ring of polynomials in ∇ with coefficients in R, and $R(\nabla)$ for the ring of rational fractions in ∇ with coefficients in R.

[3]See [17][118]. An *ideal* $\mathcal{I}$ is an additive sub-group of a commutative, integral ring $\mathcal{R}$, which is invariant under multiplication by elements of $\mathcal{R}$. It is principal if it is generated by a single element ($\mathcal{I} = a\mathcal{R}$). $\mathcal{R}$ is a principal ideal domain (*PID*) if any ideal of $\mathcal{R}$ is principal. In this case, the submodule $\mathcal{V}$ of $\mathcal{R}^n$ is $\mathcal{R}$-closed if there is a submodule $\mathcal{W}$ such that $\mathcal{V} \oplus \mathcal{W} = \mathcal{R}^n$. The closure $\overline{\mathcal{V}}$ of a submodule $\mathcal{V}$ of $\mathcal{R}^n$ is defined by $\overline{\mathcal{V}} = \{x \in \mathcal{R}^n, \exists a \in \mathcal{R}, a \neq 0, ax \in \mathcal{V}\}$. Note that $\overline{\mathcal{V}} \supset \mathcal{V}$ while $\dim \overline{\mathcal{V}} = \dim \mathcal{V}$.

Several results are obtained in modelling [17], stabilization [57], controllability [129] and observability [121] indexes, decoupling control [130] and disturbance rejection [18][19]. It is to be noted that any polynomial feedback $u(t) = -\mathbf{F}(\nabla)x(t) + v(t)$, $\mathbf{F} \in R[\nabla]^{m\times n}$ ensures the resulting system to stay in the class (13).

As far as systems with non-commensurate (but constant) delays are concerned, it is also possible to use the same approach by considering the ring of polynomials in several delay operators ∇_1, ∇_2,..., *i.e.* , $R[\nabla_1, \nabla_2, ...]$. But, the situation in this case is more complex since the ring is no longer a PID. For distributed delays, some convolution operator [66] or a ring of distributions [157] are to be introduced, with an additional complexity (indeed, it seems preferable to extend these polynomial models to rational, realizable ones, as described in the next section).

However, such a *polynomial* class of control laws appears to be limited for several advanced controllers whose concrete realizations need either *rational* fractions (precompensators by state or output feedback [118][119], neutral and 2-D systems [34][158][120]) or *distributed* delays (finite spectrum assignment [154], as we shall see). These lacks will be filled up by the generalized, rational and algebraic types of models, to be presented in the two following subsections.

2.5. Systems over rational rings. In [118][119][120] the above-mentioned realizability of concrete controllers was emphasized, by working with matrices $\mathbf{A}, \mathbf{B}, \mathbf{C}$ defined over the larger subring $\mathcal{R}_u(\nabla)$ of the irreducible, rational fractions in ∇, whose denominator has a non-zero constant term:

$$\mathcal{R}_u(\nabla) = \{n(\nabla) = p(\nabla)/q(\nabla) \in R(\nabla),\ q(z=0) \neq 0\}. \tag{15}$$

$\mathcal{R}_u(\nabla)$ coincides with the ring of proper fractions in ∇^{-1}[34][158], but also with *non-anticipative* operators[4]. For example, $\frac{1}{1+\nabla}$ (*i.e.*, $y(t) = -y(t-\delta) + u(t)$) belongs to $\mathcal{R}_u(\nabla)$, while $\frac{1}{\nabla}$ (*i.e.*, $y(t) = u(t+\delta)$) does not. This is expressed in the following properties [119]:

DEFINITION. *The transfer matrix $M(s,\nabla) \in R(s,\nabla)^{p\times m}$ is said to be causal if it has a realization over $\mathcal{R}_u(\nabla)$, i.e. if there exist matrices $\mathbf{A}(\nabla)$, $\mathbf{B}(\nabla)$, $\mathbf{C}(\nabla)$, $\mathbf{D}(\nabla)$ defined over $\mathcal{R}_u(\nabla)$ such that*

$$\begin{aligned} \dot{x}(t) &= \mathbf{A}(\nabla)x(t) + \mathbf{B}(\nabla)u(t), \\ y(t) &= \mathbf{C}(\nabla)x(t) + \mathbf{D}(\nabla)u(t), \\ M(s,\nabla) &= \mathbf{C}(\nabla)(sI - \mathbf{A}(\nabla))^{-1}\mathbf{B}(\nabla) + \mathbf{D}(\nabla). \end{aligned} \tag{16}$$

THEOREM 2.1. *The transfer function $n(s,\nabla) = p(s,\nabla)/q(s,\nabla) \in R(s,\nabla)$, with $p(s,\nabla) = p_0(\nabla) + ... + s^r p_r(\nabla)$ and $q(s,\nabla) = q_0(\nabla) + ... + s^k q_k(\nabla)$ is causal if and only if it belongs to the ring $\mathcal{R}_c \subset R(s,\nabla)$ defined by:*

1) *$n(s,\nabla)$ is s-proper (i.e., $r \leq k$),*
2) *$q_k(\nabla) \in R_u(\nabla)$.*

The advantage of a model over $\mathcal{R}_u(\nabla)$ is that any dynamic feedback law also defined over $\mathcal{R}_u(\nabla)$ keeps the resulting system in the same class (whereas $R[\nabla]$-polynomial system (13) changes of class and becomes $\mathcal{R}_u(\nabla)$-rational). Note that systems over $\mathcal{R}_u(\nabla)$ are generally neutral (see also Section 2.7).

[4] *Causality* is equivalent to *properness* defined in [34][158], which transfers were formulated with $z = \nabla^{-1}$.

2.6. Algebraic formalism: Laplace transform models for distributed delays. As in the previous case, we consider in this part linear models with commensurate delays. It is well known that the discrete-delay effect, denoted $\nabla(f)(t) = f(t-\delta)$ in the previous section, leads to the operator $e^{-s\delta}$ in Laplace transform. Then, algebraic formalism is near to the previous one $\mathcal{R}_u(\nabla)$, but explicitly considers this relation between the derivative s and delay $e^{-s\delta}$ operators. In 1985, Kamen, Khargonekar and Tannenbaum [68] introduced the set $\mathcal{G}$ of the realizable, distributed-delays transfers in which Laplace transforms can be expressed as rational functions of s and $e^{-s\delta}$. In short, $\mathcal{G} = \left\{\mathcal{L}(\text{distributed delay}) \in R(s, e^{-s\delta})\right\}$[5]. Brethé and Loiseau [10, 11, 92, 123] recently characterized this set $\mathcal{G}$ in a complete way and defined an other set, the ring of the so-called *pseudo-polynomials* (because they are analytic functions), $\mathcal{E} = R[e^{-s\delta}] \cup \mathcal{G}$, which is isomorphic to the quasi-polynomials ring $R[s, e^{-s\delta}]$. For instance, $F(s)$ is the Laplace transform[6] of the distributed transfer

$$
\begin{aligned}
& u \to y, \qquad y(t) = \int_{h_1}^{h_2} f(\theta)u(t-\theta)d\theta, \\
(17) \qquad & \frac{Y(s)}{U(s)} = F(s) = \int_{h_1}^{h_2} f(\theta)e^{-s\theta}d\theta
\end{aligned}
$$

(or, the zero-holder operator $\frac{1-e^{-sh}}{s}$ is obtained with $h_1 = 0, h_2 = h$ and the kernel $f(\theta) = 1$ over $[0, h]$, and $f(\theta) = 0$ elsewhere). The main result is that $\mathcal{E}$ is a Bézout domain [11], whose interest for finite-spectrum assignment will be emphasized in the section "Control". Note that analogous conclusion was simultaneously obtained [47] on the basis of systems over $R[s, e^{-s\delta}, e^{s\delta}]$.

2.7. 2-D models and neutral systems. It was noted [34][158] that 2-D models can be used for the modelling and control of delay systems. The Roesser models (1975) describe a two-operators system:

$$
(18) \qquad
\begin{aligned}
sX &= A_0X + A_2Z + B_0U, \\
\omega Z &= A_3X + DZ + B_3U, \\
Y &= C_1X + C_2Z.
\end{aligned}
$$

Here, (s, ω) respectively correspond to derivation and h-advance operators. If one considers, for instance, $A_2 = I$, $A_3 = A_1 + DA_0$, $B_3 = B_1 + DB_0$, then (18) corresponds to the neutral system:

$$
(19) \qquad \dot{x}\,(t) - D\,\dot{x}\,(t\text{-}h) = A_0x(t) + A_1x(t\text{-}h) + B_0u(t) + B_1u(t\text{-}h),
$$

which is a special case of (36). Such Roesser models, in turn, allow previous realization results [34] to be used for stabilization [158], and some matrix factorizations for model matching [89][90]. Equivalence with the question of realization over $\mathcal{R}_u(\nabla)$ was shown in [118][120].

2.8. Rational and finite-horizon approximations. The most common approach for control of time delay systems has been the approximation by some rational, then finite-dimension approximations, generally based on the truncation of some infinite series. Such estimations are generally inappropriate for time-varying delays. It

[5]Initially, the introduced set Θ was defined by Laplace transforms of the distributed-delays with Laplace transform in the set $R(s)[e^{-sh}]$, but in this case there is an isomorphism with $\mathcal{G}$ [10].

[6]in fact, finite Laplace transform is used in this case [98].

can be achieved by methods such as the well known Padé approximations [13][84], Hankel operator methods for infinite dimensional systems [67][46], Laguerre-Fourrier series [115][85][37] or spline approximations [4]. A case study is given in [46] on the basis of L_∞ error. However, two specific, linked problems arise with that kind of simplification: together with the problem of choosing the truncation order (hence, the dimension of the approximation), it is very difficult to prove the stability of a closed loop on the basis of such a reduced model [128][56].

The subsystems description [114][142] is another way to achieve a finite-dimensional model: in the case of a system with commensurate delays, for instance (8), the approximation is made by considering a *finite time horizon*. For any variable $z(t)$, one denotes $z_i(\theta) \triangleq z(\theta + i\delta)$ and $Z_i(\theta) \triangleq [z_0(\theta), z_1(\theta), ..., z_i(\theta)]^T$. $X_i(\theta)$ is the variable of subsystem (S_i), available on the time-interval $[0, \delta]$, with increasing size $i + 1$. The behavior of (8) on the time interval $[0, (i + 1)\delta]$ is described by:

$$
\begin{aligned}
\dot{X}_i(\theta) &= \underline{A}_i X_i(\theta) + \underline{B}_i U_i(\theta) + \underline{R}_i \Phi_i(\theta), \\
Y_i(\theta) &= \underline{C}_i X_i(\theta), \quad \text{for } \theta \in [0, \delta], \\
\underline{M}_i &= \begin{pmatrix} M_0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ M_i & \cdots & M_0 \end{pmatrix}, \quad \begin{array}{l} M = A, B, \text{ or } C \\ M_i = 0 \text{ if } i > k \end{array} \\
\underline{R}_i &= \begin{pmatrix} A_1 & \cdots & A_{i+1} \\ \vdots & \cdot & \vdots \\ A_{i+1} & \cdots & A_{2i+1} \end{pmatrix}, \Phi_i(\theta) = \begin{pmatrix} \varphi(\theta\text{-}\delta) \\ \varphi(\theta\text{-}2\delta) \\ \vdots \end{pmatrix}
\end{aligned}
$$

This technique is linked in its principle to the previously mentioned, step-by-step procedure. It was mainly used in [114][142] for studying controllability properties. However, it appears to be limited because of the size of involved matrices, which becomes larger and larger as time increases. This implies continuity problems in the junction of trajectories from model (S_i) to (S_{i+1}).

3. Stability of delay systems. Delays are reputed to destabilize the control loops. Indeed, system (2) showed that, for constant initial function $\varphi(\theta) \equiv 1$, the delay h is a source of oscillations or instability in the time response (replacing it by 0 makes oscillations disappear, whereas $h > \frac{\pi}{2}$ leads to instability). But, on the contrary, the following example

$$
\ddot{y}(t) + \omega_0^2 y(t) - k y(t - h) = 0, \tag{20}
$$

shows that delay can also have some stabilizing effect: if, in (20), h is zero, the system is oscillating or unstable. However it was noted [1] that some values of $h > 0$ and $k > 0$ make the system converge to zero (see Figure **??**).

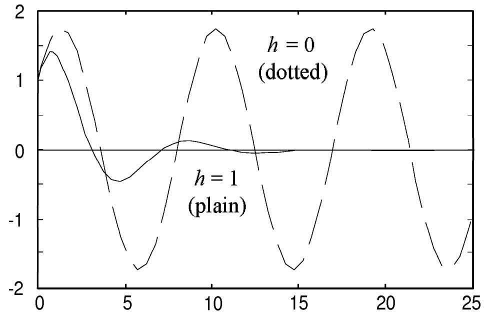

Figure 3: system (20) with $\omega_0^2 = k = 1$.

Obviously, the ability to analyze the stability of a process is a basic need for the validation of any closed-loop controller. For instance, let us mention the following, related aspects:

Asymptotic stability: do the solutions converge toward the operating point, for sufficiently small initial perturbations? This is the basic, local, asymptotic stability property.

Robustness with regard to the parameters: what are the admissible bound-values of the (constant or varying) parameters that assure the convergence? For delay systems particularly, the question is to know the maximal values of the delays (and, sometimes the minimal one) that keep the stability property. If this bound is infinite, the process exhibits the strong property of independent-of-delay stability (*i.o.d. stability*). But the assumptions for i.o.d. stability, too, may be very strong in practice and it may be preferable to look for delay-dependent conditions (*d.d. stability*), as soon as the user has information about the possible ranges of the delay variations.

Stability domains with regard to the variables: what set of initial conditions will make the state definitely converge towards the equilibrium? This question may be meaningless in linear conditions of behavior, but becomes crucial for wide-range, nonlinear models: in this last case, an answer to this question is necessary to provide the admissible changes of operating points, or for determining whether bounded additive perturbations on the state may destabilize the closed loop system.

Guaranteed, exponential decreasing rate: what is the exponential rate of convergence, this means, the velocity of the final controlled process? This point aims to compare the behavior with a first order, ordinary system: it is related to $\alpha-$stability (see below the definition).

Positive invariance: how can one be sure that a trajectory will not leave of a predetermined domain ? Such constraints may be introduced on the state (for physical security reasons), or on the control variables (for energy-limiting considerations).

In this section, some stability analysis methods are given with illustrative examples. Starting with some mathematical background on the stability of FDEs, we then propose a classification of the corresponding methods: the first part applies to linear models (foundations, frequency-domain and root-locus methods, matrix-based methods, complex plane methods, time-varying aspects); a second part deals with time-domain approaches, that are applicable to both linear and nonlinear models (first Lyapunov method, Lyapunov-Krasovkii functionals, Lyapunov-Razumikhin functions, comparison methods). The reader can find some more complete presentations in the previously mentioned books as [76][33] or also in [22]. As for ODEs, the stability property is classically defined for system (3) in free motion ($u \equiv 0$), that is supposed to have a unique solution, with an equilibrium solution at zero:

$$
\begin{aligned}
\dot{x}(t) &= f\left(x_t, t\right), && t \geq t_0, \\
x(\theta) &= \varphi(\theta), && t_0 - h \leq \theta \leq t_0, \\
f(0,t) &= 0, && \forall t.
\end{aligned}
\tag{21}
$$

The solution is denoted $x(t; t_0, \varphi)$ or, briefly, $x(t)$. Mainly, the concepts are the same as for ODEs, but replace the norm of initial values by some uniform norm of function $\|\varphi\| \triangleq \max{}_{-h\square\theta\square 0} \left|\varphi(\theta)\right|, |x|$ denoting a norm of vector x.

DEFINITION. *The zero solution of system (21) is:*

1) *Stable if for any* $\varepsilon > 0$ *and any* t_0 *there exists* $\delta = \delta(\varepsilon, t_0) > 0$ *such that* $|x(t; t_0, \varphi)| < \varepsilon$ *for all* $\varphi \in \mathcal{C}$ *verifying* $\|\varphi\| < \delta$ *and for all* $t \geq t_0$.

2) *Asymptotically stable if it is stable and if, for any solution* $x(t)$ *of the stability problem* 1), *we have* $\lim_{t\to\infty} |x(t)| = 0$.

3) *Uniformly, asymptotically stable if* 1) *holds with* $\delta = \delta(\varepsilon)$ *and* 2) *holds with uniform limit* $(t \to \infty)$.

4) $\alpha-$*stable if* 3) *holds with the more constraining condition* $\lim_{t\to\infty} |e^{\alpha t}x(t)| = 0$, *with* $\alpha > 0$. *The constant* α *is called a guaranteed (exponential) decay rate of (21).*

5) *Uniformly, asymptotically stable independent of the delays (shortly, i.o.d. stable) if it is uniformly, asymptotically stable for all positive values of the upper bound* h.

In engineering practice, the parameters of a model are known with a finite precision only; then, the model can be considered as the sum of two terms:

$$\dot{x}(t) = f\left(x_t, t\right) + \Delta f\left(x_t, t\right)$$

where the first part f represents the nominal model, and the second part Δf represents the uncertainties on the model. All we know about this second term is that it belongs to a certain set of functionals $\mathcal{D}$. Generally, the nominal model is linear and uncertainties are described by their bounds: for instance, in the case of *unstructured uncertainties*, $\mathcal{D}$ is the set of continuous functions such that $\left\|\Delta f\left(x_t, t\right)\right\| \leq \delta \left\|x_t\right\|$ (norms), and for *structured uncertainties*, $\left|\underline{F}x_t\right| \leq \left|\Delta f\left(x_t, t\right)\right| \leq \left|\overline{F}x_t\right|$ (absolute values) with linear mappings $\underline{F}$ and $\overline{F}$. This yields the following definition.

DEFINITION. *The zero solution of system (21) is robustly (asymptotically) stable with regard to set* $\mathcal{D}$ *if it is (asymptotically) stable for any* $\Delta f \in \mathcal{D}$.

3.1. Basic stability property in the linear case. In relation to the stability of the linear equations (6), the necessary and sufficient condition (N.S.C.) is also a straightforward generalization of ODE's theory, based on the research of some particular, nontrivial, exponential solution $x(t) = e^{st}$.

THEOREM 3.1. *The zero equilibrium of retarded system (6), with* $C_j(s) = C_j$ *constant and* $D_k = 0$, *is asymptotically stable if and only if all the zeros (s) of the characteristic function (22),* $p(s)$, *have negative real parts,*

$$p(s) = \det\left(sI_n - A_0 - \sum_{i=1}^{m} A_i e^{-sh_i} + \sum_{j=1}^{r} C_j \frac{1 - e^{s\tau_j}}{s}\right). \tag{22}$$

Note that in this retarded case, there can only be a finite number of unstable roots, which is no longer true in the neutral case: if $D_k \neq 0$, the term $-s\sum_{i=1}^{k} D_k e^{-s\omega_k}$ is to be added inside the determinant. In the complex plane, there may be infinite branches of roots tending to the imaginary axis: conditions based on the sign of the real parts must then be considered with great care [74]. But, assuming the stability of the difference equation $x(t) + \sum_{i=1}^{k} D_k x(t - \omega_k) = 0$ (see (36)), it holds that the number of unstable roots is finite [35].

Then, checking such conditions is much harder than in the ODE's case: $p(s) = q(s, e^{-s})$ is not a polynomial in s and there is no equivalent to the Routh-Hurwitz test. Hand-calculating the characteristic roots of the very simple, scalar example

$$\dot{x}(t) = -ax(t) - bx(t-h), \quad x \in R, \tag{23}$$

($s + a + be^{-hs} = 0$) illustrates how difficult it can be to carry a direct analysis of the transcendental equation (22) for systems with certain dimension, or for designing some tuning parameters. We shall see in the following that there are many stability criteria, but none of them gives necessary and sufficient conditions which are simple and practical at the same time[7]. These methods are presented in a synthetic way in [22][27][108][33].

As a general observation concerning delay-dependent and i.o.d. criteria, it is worth noticing that, when applied to the prototypical system

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-h), \quad x \in R^n, \tag{24}$$

the first class often needs the matrix $A_0 + A_1$ to be Hurwitz, while the second, of course more constraining, demands this condition for A_0. Lastly note that the consideration of multiple delays is accompanied by a huge increase of computational complexity (see an evaluation in terms of *NP*-hardness in [140]).

3.2. Linear systems: frequency-domain, root-locus. Then, the stability analysis of a linear, time-invariant system with delays is grounded in its characteristic equation (22). Of course, as the characteristic function $p(s)$ depends on the delay, a system may be stable for some set $\{\omega_k, h_i, \tau_j\}$ and unstable for an other set. The extensions of Routh-Hurwitz criterion proposed by Pontryagin (1942) or Chebotarev (1949) [76] are seldom applicable in practice. Besides, the rational approximations (as Padé's ones) are not very relevant, since the study has to be carried up to an undetermined order. Fortunately, there are some interesting methods that allow the analysis of the characteristic equation in a necessary and sufficient way: in addition to the Tsypkin i.o.d. stability criterion[8], let us mention here the Pontryagin method (for commensurate delays), the D-partition approach (dividing the space of the parameters into several regions, which boundaries correspond to critical stability), the methods by τ-partition (for commensurate delays, dividing the study on intervals of delays) as, in particular, the interesting method of Walton and Marshall (1987, commensurate delays, polynomial analysis) or the similar, pseudo-delay approaches (Rekasius 1980, Hertz, Jury and Zeheb 1984, 1987) and methods by Kamen (1980-1983, commensurate delays, i.o.d. conditions, methods based on 2-variable polynomials in (s, z), $z = e^{-\delta s}$).

The general drawback of these necessary and sufficient conditions, restricted to constant delays, is that they are difficult to apply when several parameters are to be

[7] However, for (23), explicit N.S.C. are known: (23) is asymptotically stable for any value h iff $a + b > 0$ and $a \geqq |b|$; it is asymptotically stable for any value of h less than h^* if $b > |a|$ and $h^* = (b^2 - a^2)^{-1/2} \arccos(-a/b)$.

[8] Restricted to single-delay, open-loop-stable transfert functions $\frac{e^{-hs} P(s)}{Q(s) + e^{-hs} P(s)}$, the Tsypkin N.S.C. demands polynomials P (degree $n-1$) and Q (stable, degree n) to satisfy $|P(j\omega)| > |Q(j\omega)|$ for all $\omega \in R$.

tuned. Lastly, we want to mention the Chebotarev method, whose theory needs to check an infinite number of determinants but, conversely, can be used as a necessary condition of stability. Description and examples are given in [76][22][27][108][33].

3.3. Linear systems: methods in the complex plane. Classical stability conditions such as Nyquist or Mykhailov-Leonhard criteria can easily be generalized for systems with delays. Indeed, the *argument principle*, central core of these criteria, is still applicable since the number of the unstable roots in the complex plane is finite. The induced methods [76][22][108][33] generally apply, in a necessary and sufficient way, to constant but non necessarily commensurate delays. They yield computational difficulties when many combinations are to be checked, with complex parameters.

3.4. Linear systems: matrix-based methods. Several results are expressed in terms of sufficient (but non necessary) conditions, involving the matrix measures and norms[9]. Compared with previous frequency approaches, this lack of necessity is compensated by the relative ease of implementation. We shall see in a coming section, devoted to the comparison approach, that some of these approaches may also hold for nonlinear models. Among the various methods, let us just recall here the very representative Mori, Fukuma and Kuwahara criterion (1981)[102] and previous result by Tokumaru *et al.* 1975, for single-delay systems, which further gave rise to generalized formulations (for instance, Brierley *et al.* 1982, Hmamed 1986, Mori and Kokame 1989, Dambrine and Richard 1993, Kolmanovskii 1995, Goubet *et al.* 1997). These other statements can be found, with examples, in [22][108] [33].

THEOREM 3.2. *The system (24) is i.o.d. stable if* $\mu(A_0) + \|A_1\| < 0$. *Moreover, its solution verifies* $\|x(t;0,\varphi)\| \leq \|\varphi\|\, e^{-\sigma t}$ *(*$t \geqslant 0$*), where* σ *is the real solution of equation* $1 + \frac{\sigma}{\mu(A_0)} + \frac{\|A_1\|}{\mu(A_0)} e^{\sigma h} = 0$.

Other results for commensurate-delays systems, by Chen (1994) and Su (1995), involve generalized eigenvalues[10] and matrix pencils techniques [111]. They need to check matrices of increased order, obtained by sums and products of Kronecker. Delay-dependent or i.o.d. criteria can be obtained (see [33]). The main difficulty here is the high-dimensional computations of large-scale pencils (the dimension multiplies with the number of delays).

3.5. Linear, time varying systems. Except some matrix-based methods, the previous results do not apply anymore if the delay is time-varying. The following example, with $T = 1$, $a = 3.5$, $b = 4$, has been shown [64] to be unstable:

$$\begin{aligned} \dot{x}(t) &= -ax(t) - bx(t - h(t)) \\ h(t) &= t - kT, \ \forall t \in \,]kT, (k+1)T] \quad \text{(then } h(t) \leq T) \end{aligned}$$

while for any constant value of the delay $h(t) \equiv h \leq 1$, its characteristic roots have negative real parts. Inversely, for $a = -1$, $b = 1.5$, it is asymptotically stable, while linear time-invariant conditions don't hold (see also [22], and note that this kind of delay variation -Fig. 4- corresponds to a T-periodic sampling).

[9] The measure (or logarithmic norm) $\mu(A)$ of a matrix A, associated to a norm, is $\mu(A) = \lim_{\varepsilon \to 0} \frac{\|I+\varepsilon A\|-1}{\varepsilon}$; matrix norm is $\|A\| = \sup_{x \in R^n} \frac{\|Ax\|}{\|x\|}$. Measure may be negative, norm must be nonnegative.

[10] A generalized eigenvalue of matrices A and B is a complex number λ such that $det(A - \lambda B) = 0$ (the number of finite generalized eigenvalue is at most equal to the rank of B).

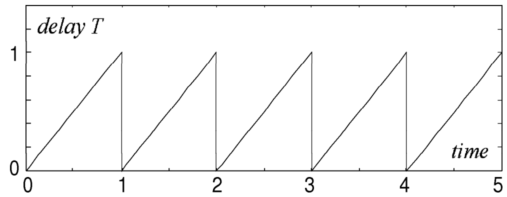

Figure 4: time-varying delay in T-periodic sampling situation.

Then, even if the works which take into account time-varying delays are fewer, they are of practical interest when designing the control of a process whose delay variations are actually non negligeable [93] [148][96]. The following methods (presented for nonlinear systems) have then to be developed and used for this time-varying case, as for the classical ODEs. For instance, the simple system (2) was shown to be asymptotically stable for time-varying $h(t)$ if $h(t) \leq h < 1$ (note that this condition is only a sufficient condition). The special case of linear, periodic-time varying delay systems (*i.e.* (7) in free motion with periodic A_i, h_i) received particular attention, with the generalization of the monodromy operators and *characteristic multipliers* encountered in the Floquet-Lyapunov theory (Stokes 1962, Halanay 1966, see [60][76][22]): but, here again, delays imply an increasing complexity.

3.6. Nonlinear systems: the first Lyapunov method. The first Lyapunov method [35] still holds for the system

$$\begin{aligned} \dot{x}(t) &= \sum_{i=0}^{k} A_i x(t-h_i) + q(t, x_t) \\ q(t,x_t) &= q(t, x(t), x(t-\tau_1(t)), ...x(t-\tau_k(t)) \\ h_0 = 0,\ h_i &= \text{constant}, \quad \tau_j(t) \in [0, \tau_i] \text{ continuous}, \end{aligned} \tag{25}$$

with a function q such that, for any u_i, $\|u_i\| \leq \varepsilon \Rightarrow \|q(t, u_0, ..., u_k)\| \leq \beta_\varepsilon(\|u_0\| + ... + \|u_k\|)$, with constant β_ε uniformly decreasing to 0 as $\varepsilon \to 0$. The "tangent", linearized system is, as usual, defined by

$$\dot{x}(t) = \sum_{i=0}^{k} A_i x(t-h_i). \tag{26}$$

THEOREM 3.3. *If system (26) is asymptotically stable, then the zero solution of (25) is, too. If (26) has at least one characteristic root with positive real part, then the zero solution of (25) is unstable.*

This result can be followed by some small-delays approximation theorem (obtained by continuity of the characteristic roots with regard to delays h_i). Here, "small" is to be understood as "sufficiently small".

THEOREM 3.4. *If* $\sum_{i=0}^{k} A_i$ *is a Hurwitz matrix, then the zero solution of (26) is asymptotically stable for small values of the delays* h_i. *If this matrix is unstable, then the zero solution of (26) is unstable for small values of the delays* h_i. *If* 0 *is a single eigenvalue of this matrix, the other having negative real parts, then the zero solution of (26) is stable for small values of the delays* h_i.

3.7. Nonlinear/linear systems: time-domain methods. The next subsections present three approaches based on time-domain, FDEs representations. The direct method of Lyapunov has been extended to FDEs in two different ways: the first one, due to Krasovskii (1963), uses a *functional* generalization of the notion of Lyapunov function; the other one (Razumikhin, 1956) keeps the classical approach of Lyapunov *functions* but applies it to a certain type of solutions. For further details, the reader can refer to [76][108][33]. A third approach is based on *comparison* systems, and will be presented last (see details in [22][49][33]).

These very general time-domain approaches apply to both linear and nonlinear systems:

- *in the linear case*, they contribute to many results on *robust stability*, whereas previous necessary and sufficient conditions are quite limited (see a survey of the induced, sufficient conditions in the first chapter of [33]);

- *in the nonlinear case*, they simply appear as the only way to acheive the stability analysis.

3.8. Functional approach of Lyapunov-Krasovskii. This section gives a short overview of the first class of the above-mentioned, time-domain methods. In order to extend the Lyapunov's direct method to FDEs, Krasovskii (1963) proposed to consider functionals instead of classical Lyapunov functions. This generalization permits in particular the obtension of some converse theorems. It is based on the following, classical result:

THEOREM 3.5. *System (21) is asymptotically stable if there exists a continuous functional $V(t,\varphi) : R\times \mathcal{C} \longrightarrow R^+$, which is positive-definite, decrescent, admitting an infinitesimal upper limit, and whose full derivative $\dot{V}(t,x_t)$ along the motions of (21) is negative definite over a neighborhood of the origin.*

Among the particular choices of the functional V, several authors proposed stability conditions for linear systems (24) with the following, "generalized quadratic form"

$$V(x_t) = x(t)^T P x(t) + \int_{-h}^{0} x(t+\theta)^T S x(t+\theta) d\theta.$$

This functional, applied to the linear systems, leads to sufficient conditions in the form of Riccati equations, as follows (see for instance [108]).

THEOREM 3.6. *System (24) is i.o.d. stable if there exist positive-definite, symmetric matrices P, S, R verifying the following, auxiliary Riccati equation*

$$A_0^T P + P A_0 + P A_1 S^{-1} A_1^T P + S + R = 0. \tag{27}$$

Other i.o.d. conditions [131][33] were formulated in terms of Riccati equations. More complex functionals lead to delay-dependent conditions, available for discrete-single [112], discrete-multiple [33][78] and distributed [149][77] delays.

Moreover, many such Riccati-type results were translated in terms of linear matrix inequalities (LMIs, see [9]) [33][112][109][108][75]. For instance, condition (**??**) can be equivalently checked by means of LMIs, as

$$\begin{pmatrix} A_0^T P + P A_0 + S & P A_1 \\ A_1^T P & -S \end{pmatrix} < 0.$$

Note that the major part of the delay-dependent conditions were obtained by using some other formulation of the initial, generally linear system. For instance, the following system,

$$\dot{x}(t) = \sum_{i=1}^{m} A_i x(t - h_i). \tag{28}$$

can be written under the three following forms [78],

$$\dot{x}(t) = Ax(t) - \sum_{i,j=1}^{m} A_{ij} \int_{t-h_{ij}}^{t-h_j} x(s)ds, \tag{29}$$

$$\dot{x}(t) = Ax(t) - \sum_{i=1}^{m} A_i \int_{t-h_i}^{t} \dot{x}(s)ds, \tag{30}$$

$$\frac{d}{dt}\left[x(t) + \sum_{i=1}^{m} A_i \int_{t-h_i}^{t} x(s)ds\right] = Ax(t), \tag{31}$$

with notation

$$A = \sum_{i=1}^{m} A_i,\ \ A_{ij} = A_i A_j,\ \ h_{ij} = h_i + h_j,\ \ h = \sum_{i=1}^{m} h_i.$$

Each formulation can be studied by specific Lyapunov-Krasovskii functionals, leading to the three different Riccati equations [78],

$$A^T P + PA + mRh + P \sum_{i,j=1}^{m} h_i A_{ij} R^{-1} B_{ij}^T P = -Q, \tag{32}$$

$$A^T P + PA + \sum_{i=1}^{m} (h_i P A_i R^{-1} B_i^T P + mhA_i^T R A_i) = -Q, \tag{33}$$

$$\begin{aligned} -Q &= A^T P + PA + \\ &\sum_{i=1}^{m} R_i h_i + \sum_{i,j=1}^{m} A^T P A_i R_i^{-1} A_i^T P A h_i. \end{aligned} \tag{34}$$

Then, the system (28) is asymptotically stable if for some symmetric positive matrices R_i and Q there exists a positive solution, P, of one of the equations (32), (33), (34). The resulting conditions depend on *all* the delays values, but dependency can be reduced to some *chosen* delays [75].

However, in the general, nonlinear case, finding a suitable functional V can be compared to... an *art*! [14][73] This question, already encountered with ODEs models, is even more pertinent for FDEs. A formal procedure to construct Lyapunov functionals V for concrete equations with delay was proposed by Kolmanovskii [73][71]. Basic features of this procedure are as follows: represent the right-hand side of the equation as a sum of two terms, first of which has the form of an instantaneous negative feedback; construct a Lyapunov function v for the auxiliary ordinary differential equation corresponding to the first term; obtain functional V from by changing the arguments of v. Note that various steps of the procedure can be implemented non-uniquely.

3.9. The Lyapunov-Razumikhin approach. Because of the complexity of the construction of a Lyapunov functional for nonlinear models, Razumikhin (1956) proposed another generalization of Lyapunov second method, keeping the idea of Lyapunov *functions* $V(x(t))$ (and not functional $V(x_t)$). The great difference is that the derivative of the chosen Lyapunov function has to be negative only for special solutions of the system (very roughly speaking, the idea is to check the sign of the derivative $\dot{V}$ only when the state function may leave a set $V(x) =$ constant, see Fig. 5).

THEOREM 3.7. *Let $u(\rho)$, $v(\rho)$, $w(\rho)$ and $p(\rho)$ $(R^+ \to R^+)$ be continuous, nondecreasing functions, positive for $\rho > 0$, $u(0) = v(0) = 0$ and $p(\rho) > \rho$ for $\rho > 0$. If there is a continuous function $V : R \times R^n \to R$ such that $u(\|x\|) \leq V(t,x) \leq v(\|x\|)$ for any (x,t), and $\dot{V}\ (t,x(t)) \leq -w(\|\ x(t)\ \|)$ for states x_t verifying $\{\forall\theta \in [-h,0], V(t+\theta, x(t+\theta)) < p(V(t,x(t)))\}$, then the zero solution of (21) is uniformly asymptotically stable.*

A practical corollary was given in [150], changing the last condition into: $\dot{V}$ $(t,x(t)) \leq -w(\|\ x(t)\ \|)$ *for states* x_t *verifying* $\forall\theta \in [-h,0]$, $\|\ x(t+\theta)\ \| < \eta \ \|\ x(t)\ \|$ *for an* $\eta > 1$.

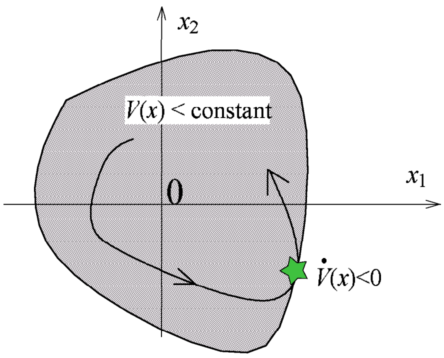

Figure 5: principle of the Razumikhin's theory.

3.10. The LaSalle principle. The invariance principle of LaSalle (1960) is a well known extension of the Lyapunov functions theory, that allows one to study of asymptotic behavior of ODEs solutions (in particular, the boundedness properties). It involves the notion of positive invariance of sets, that can be easily generalized to FDEs. Then, the LaSalle invariance principle was extended to retarded, time-invariant systems, by using either the Krasovskii functional procedure (Hale, 1965 [60]) or the Razumikhin functions one [58]. We don't give the statements here (see for instance [22]).

3.11. The comparison approach. The direct stability analysis of a complex system often remains too cumbersome or can even be impossible to perform. An alternative, indirect way is to proceed *via* a simpler system, called *comparison system.* This notion was originally defined for ODEs [100] and then, extended to FDEs [83]. Firstly, we present a wide definition of the idea.

DEFINITION. *A system $(\mathcal{A})$ is said to be a comparison system of a system $(\mathcal{B})$ with regard to the property $\mathcal{P}$ (for example, stability of its zero solution), if the verification of property $\mathcal{P}$ for system $(\mathcal{A})$ implies the same property for system $(\mathcal{B})$.*

For instance, the first-order approximation of a nonlinear ordinary differential equation may be viewed as a comparison system with regard to the local, uniform asymptotic stability. However, most comparison systems rely on differential inequalities [83][71] and vector-Lyapunov functions[11] [23][25][22][51], whose tools constitute the framework of the approach. The major part of the referenced results use a Razumikhin approach in their proof. The next definition is a continuation of the previous one.

DEFINITION. *Let* $V : R^n \to R^k_+$ *(with* $k \le n$*) be a continuous, positive function such that* $V(x) = 0 \Leftrightarrow x = 0$. *Assume that, along the solutions of (21), the right-hand time-derivative (Dini derivative) of* $y(t) = V(x(t))$ *satisfies the functional differential inequality* $D^+y(t) \le \mathcal{F}(t, y_t)$. *Then system* $\dot{z}(t) = \mathcal{F}(t, z_t)$ *is an overvaluing system of (21) with respect to the function* V *if the inequality* $V(x(t)) \le z(t)$ *holds for any* $t \ge t_0$ *as soon as it holds for initial times* $t \in [t_0 - h, t_0]$.

Using the assumptions made on V, it is simple to prove that an overvaluing system is also a comparison system with regard to stability or asymptotic stability. Conditions on functional $\mathcal{F}$ to be an overvaluing system are called comparison principles (as, in ODEs case, the so-called Ważewski conditions): some of them, very general, are recalled in [22][33]. A particular but interesting comparison principle can be mentioned here for illustration, providing an exponential convergence rate γ (this lemma was proven in [141] for a single-delay inequality, and under this two-delay form in [51]).

THEOREM 3.8. (LEMMA). *Let* C, D_1 *and* D_2 *be* $n \times n$ *matrices with real entries and let* $x(t)$ *be a solution of the differential inequality (*$t \ge 0$*),*

$$\dot{x}(t) \le g(x_t),$$
$$g(x_t) = -Cx(t) + D_1 \sup_{0 \le \theta \le h_1} x(t-\theta) + D_2 \sup_{0 \le \theta \le h_2} x(t-\theta).$$

Assume that $D_1 \ge 0$, $D_2 \ge 0$, *that the off-diagonal entries of* C *are non positive, and that* $(-C + D_1 + D_2)$ *is the opposite of an M-matrix*[12]. *Then the solution* $x(t)$ *of this inequality is overvalued by the asymptotically stable solution* $z(t)$ *of the differential equation* $\dot{z}(t) = g(z_t)$, $t \ge 0$, *with initial condition* $0 \le x(\theta) \le z(\theta)$ *for* $h \le \theta \le 0$ *(*$h = -\max\{h_1, h_2\}$*).*

If in addition $(-C + D_1 + D_2)$ *is irreducible, then there is a constant* $\gamma > 0$ *and a constant vector* $k_\gamma > 0$ *such that* $x(t) \le k_\gamma e^{-\gamma t}$ *for* $t \ge 0$. *Here,* γ *and* k *are obtained in the following way:* γ *is the positive real solution of the equation* $\lambda_m(A_\gamma) = -\gamma$, *where* $A_\gamma = -C + D_1 e^{\gamma h_1} + D_2 e^{\gamma h_2}$. k_γ *is a positive, importance eigenvector of* A_γ *associated with the importance eigenvalue* $\lambda_m(A_\gamma)$.

Vector-norms (each entry of V is a scalar norm of a subvector x_i of x) constitute a particular case of the general (but hard to solve) vector-Lyapunov function(al)s: they lead to systematic determination of comparison systems in many cases of FDEs [22][49][33][138]. Applying this tool on a nonlinear system with a single delay h, a systematic construction of matrices C, D_1, and possibly D_2 (with $h_1 = h$, $h_2 = 2h$) is given in [51][33], leading to the following, simple conditions ($|.|$ denotes here the

[11]Vector-Lyapunov functions were simultaneously introduced for ODEs in [5] and [100].

[12]a matrix A is the opposite of an M-matrix if all its off-diagonal elements are non-negative and if A is Hurwitz. This latter condition may be easily tested by verifying that all its successive, principal minors are negative. Such matrix A has a real eigenvalue $\lambda_m(A)$ which is greater than the real parts of all others. $\lambda_m(A)$ is called the importance eigenvalue of A. If A is irreductible (*i.e.* if it is not similar to a bloc-triangular matrix), then there is an associated, importance eigenvector u_m of A verifying $u_m > 0$.

entry-to-entry absolute value of vectors or matrices, M^* denotes the matrix obtained from M by replacing all its off-diagonal entries by their absolute values).

THEOREM 3.9. *The zero equilibrium of the uncertain system*

$$
\begin{aligned}
\dot{x}(t) &= Ax(t) + Bx(t - h(t)) \\
&+ f(x(t), t) + g(x(t - h(t), t) \\
|f(x,t)| \le F\,|x|\,, &\quad |g(x,t)| \le G\,|x|\,, \\
h(t) \le h, &\quad B = B' + B''
\end{aligned}
$$

is asymptotically stable if the matrix $M = (A + B')^* + |B''| + F + G + h[|B'A| + |B'B| + |B'|\,(F + G)]$ *is Hurwitz.*

Note that it does not need A to be Hurwitz, but $A + B'$. Such a result is very closed to the matrix-based methods seen in the linear case[13]. Recently, results using the same comparison approach for discrete-plus-distributed delay systems have also been obtained [138] on the basis of transformations such as (29). This shows that some comparison results are directly workable: even if the main question with such procedure may be its non-uniqueness (dependence with regard to the chosen state basis and to the decomposition $B = B' + B''$), it provides information about both qualitative and quantitative stability aspects that were presented at the beginning of this section. For instance, comparison systems allow to estimate positively invariant sets, convergence rate, or stability domains with regard to initial conditions: this is illustrated by the system

$$
\begin{aligned}
\dot{x}\,(t) = &\begin{bmatrix} -2 + \alpha(.) & 0 \\ 0 & -1 + \beta(.) \end{bmatrix} x(t) \\
+ &\begin{bmatrix} -1 + 0.1x_2(t - h) & 0 \\ \gamma(.) & -1 + \delta(.) \end{bmatrix} x(t - h)
\end{aligned} \tag{35}
$$

with bounded, varying parameters$|\alpha(.)| < 1.6$, $|\beta(.)| < 0.5$, $|\gamma(.)| < 1$, $|\delta(.)| < 0.3$, $h \in [0, h_m]$. The results [33] are shown on Figure 6, which presents different estimates in relation with the maximum value h_m. Two simulations are also represented, inside and outside the estimated domain for $h_m = 0.01$.

[13]For unstructured perturbations $\|f(x,t)\| \,\square\, \alpha\,\|x\|\,,\ \|g(x,t)\| \,\square\, \beta\,\|x\|\,,$ the condition can be stated in terms of measures as $\mu(A + B') + \|B''\| + \alpha + \beta + h\,(\|B'A\| + \|B'B\| + \|B'\|\,(\alpha + \beta)) < 0$.

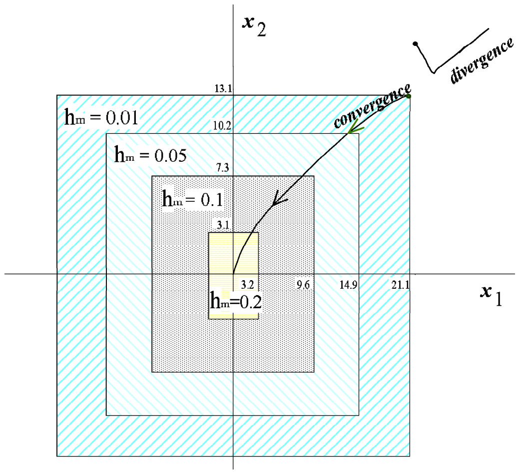

Figure 6: delay-dependent stability domains estimates of system 35.

This simplicity, compared with the wideness of the admissible models and possible applications, constitutes a key point of the comparison approach.

3.12. The case of neutral systems. The above mentioned functional approach is also fruitful for neutral systems [132][76][61][74]. In this case, the procedure can be generalized to a slightly more complex one, involving functionals $V(Fx_t)$ with notation referring to (5) and the stability of the operator F also has to be checked: F is stable if the zero solution of the equation $Fx_t = 0$ is uniformly, asymptotically stable. For instance, considering the (usual) case $Fx_t = x(t) - Dx(t-h)$, with constant matrix D, a necessary stability condition for the linear, neutral system

$$\dot{x}\,(t) - D\,\dot{x}\,(t-h) = \sum_{i=0}^{k} A_i x(t-h_i) \tag{36}$$

to be stable is that D has eigenvalues inside the unit circle (or, equivalently, is Schur-Cohn stable), whose property is also called "*formal stability*" of system (36) [16].

THEOREM 3.10. *Consider the equation*

$$\frac{d}{dt} F x_t = f(x_t), \tag{37}$$

with $f : \mathcal{C} \to R^n$ taking bounded sets of $\mathcal{C}$ into bounded sets of R^n. Suppose F is stable, and that $u(\rho)$, $v(\rho)$, $w(\rho)$ are continuous and nondecreasing functions, cancelling at $\rho = 0$ and positive elsewhere. If there is a continuous function $V : R \times \mathcal{C} \to R$ such that $u(\|F\varphi\|) \leq V(t,\varphi) \leq v(\|\varphi\|)$ and, along the motions, $\dot{V}\,(t, x_t) \leq -w(\|\,x(t)\,\|)$, then the zero solution of (37) is uniformly asymptotically stable.

On this basis, Riccati equations can be constructed [33, 132] as well. In 1979, Kolmanovskii and Nosov [76] also defined the principles of f-*stability* (generalization of the above "formal stability" to nonlinear systems) and *degenerate functionals* ($V(x_t)$ is said to be degenerate because it may be equal to zero even if the function x_t is not identically zero) for stability study of nonlinear, neutral equations. Results based on the comparison approach were given by Tchangani *et al.* [33][139], together with estimates of the stability domains and asymptotic-behaviors bounds.

3.13. Stabilization. Many studies are devoted to the stabilization of time-delay systems. The previous stability criteria are of course directly involved in such control study, but some of them are more useful depending on the kind of stabilization problem.

Concerning the linear, time-invariant models, the methods are related to the controllability properties, with a great interest in the *finite-spectrum assignment* problem. Since the stability tests are to be made on the characteristic equation (by previously presented N.S.C.), they are much simpler in the particular case of finite-spectrum assignment, since the aim is then to obtain a polynomial equation (hence, with finite number of roots). This aspect will be presented in the following section "Control".

Concerning *robust stabilization* of linear models with constant or nonlinear, time-varying parametric uncertainties (see for instance [33]), the methods are mainly based on the time-domain Krasovskii approach or on the comparison approach; both allow one to deal with time-varying delays, whereas the frequency-domain and complex-plane methods generally need the delays to be constant.

The problem of stabilization with *input-disturbances* can be treated by means of H_∞ norms: this involves time-domain approaches, mainly the Krasovskii generalized quadratic functionals, leading to Riccati equations or LMIs (see for instance [145][38] [154][108][109] and their references).

It is to be mentioned that the main part of these last two categories (robustness-type results) are dealing with systems with memoryless input (*i.e.* no delay on the control), which imposes a real restriction: the delay phenomenon is often induced by the actuators or sensors. A possible solution to this problem consists in introducing an integrator in the control: the simple system $\dot{x}\ (t) = u(t-h)$ is then transformed by $(u(t) = x_2(t))$ into:

$$\begin{aligned} \dot{x}_1\ (t) &= x_2(t-h), \\ \dot{x}_2\ (t) &= v(t). \end{aligned}$$

Lastly, *constrained stabilizing control* is mainly grounded in the positive-invariance property, which is slightly more difficult to handle in functional spaces. Several results have been obtained in the case of linear delay systems [110][52][62] and nonlinear ones [26][33].

4. Structural properties. Controllability and observability of delay systems have been studied through different modelling approaches. Compared to ODEs, two main differences arise from the presence of aftereffects (see their illustration on Fig. 7). The first is related to the state variables: instead of reaching a point at a time t_1, the actual notion of controllability means to reach a *function*, which means to assign the vector $x(t)$ from time t_1 to time $t_1 + h$. Two large classes of properties can then be distinguished:

- the *functional* properties aim to reach a function $\phi \in \mathcal{C}$ at time t, this means making the behavior reach some predetermined function $x_t \in \mathcal{C}$. These functional properties mainly correspond to the infinite-dimensional models; among them, the *spectral* properties only concern the eigenvalues, thus, problems of stabilization or observation.

- the *point-wise* properties consider the problem of reaching the point $x \in R^n$ (solution at a given time); they can be studied through all the above-mentioned classes of models.

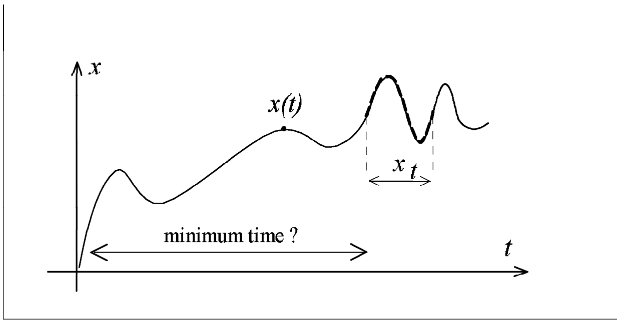

Figure 7: an illustration of the controllability questions for FDEs.

The second difference is linked with time: even if the model is linear, delays yield the existence of a required, minimum *reaching time*. In other words, the usual question of the reachable sets (obtained, for linear systems, by checking "orders" of Kalman-like controllability chains) has to be completed by associating a "class" depending on the time needed to achieve the control.

Lastly, the question of the nature of control laws to be implemented (involving static or dynamic feedbacks, with discrete or distributed delays) also constitutes an important issue: controllability properties over $R[\nabla]$ (*weak*) or over $R(\nabla)$ (*strong*) are related to this aspect.

Many authors contributed to this study of structural aspects: surveys can be found in [129][82][69]. Correspondences between different properties in a unifying framework (in the module theory) were given by Fliess and Mounier [105][40][92][42] and leads to practical applications (see for instance [117]). In this paper, we shall mainly deal with the notions related to controllability.

4.1. Functional controllability properties. Infinite-dimensional models as (11) received several controllability definitions [29][99], that are of *functional* type. The following ones refer to system (11) with solution (12), and are also called "*approximate controllability*":

DEFINITION. *The state $\overline{x}_0$ is $\mathcal{M}_2$-controllable at time t to $\overline{x}_1 \in \mathcal{M}_2([-h,0];R^n)$ if there is a sequence of controls $\{u_i\}$ defined in $\mathcal{L}_2([0,t];R^m)$ such that $\overline{x}(t;\overline{x}_0,u_i)$ converges to $\overline{x}_1$ (in the sense of the norm over $\mathcal{M}_2$). The system (11) is $\mathcal{M}_2$-controllable at time t if all states $\overline{x}_0$ are $\mathcal{M}_2$-controllable at time t to any $\overline{x}_1 \in \mathcal{M}_2([-h,0];R^n)$.*

$\mathcal{M}_2$-controllability was characterized by means of N.S.C. [29] but the conditions are not so easy to check. This notion corresponds to a restricted notion (*approximate*) if one compares it with the $\mathcal{M}_2$-*strict controllability at time t* defined in [29]: it only involves the *limit* trajectories that can be obtained by sequences $\{u_i\}$, because the domain of definition of the operator $\widetilde{A}$ is strictly included in $\mathcal{M}_2$ (only its adherence is equal to $\mathcal{M}_2$), while the strict notion needs a unique, concrete control law u to exist. Other forms of controllability were defined [99] for $\overline{x}_0 = 0$.

For systems with delayed control but without delay on the state variables, the notion of *absolute controllability* was defined (together with simple N.S.C.) in [114].

DEFINITION. *The linear system with commensurate delays (8) with the restriction $\forall i \geq 1$, $A_i = 0$ is absolutely controllable if, for any initial condition $\{x_0,\ u(t)_{t\in[-k\delta,\ 0]}\}$, there is a time $t_1 > 0$ and a bounded control law $u(t)$ such that $x(t_1) = 0$ with $u(t) = 0$ for all $t \in [t_1 - k\delta,\ t_1]$.*

THEOREM 4.1. *The system (8) with $A_i = 0$ $\forall i \geq 1$, is absolutely controllable if and only if* $\operatorname{rank}\left[E, A_0E, ..., A_0^{n-1}E\right] = n$, *with* $E = \sum_{i=0}^{k} e^{-i\delta A_0} B_i$.

Absolute controllability is actually a functional property, since it implies the ability to maintain $x(t)$ at zero on a time interval $[t_1, t_1 + k\delta]$. However, the main problem resides in its very demanding definition, needing $u(t) = 0$ for all $t \in [t_1 - k\delta,\ t_1]$: such an "ending free-motion" is too constraining in general.

Another property was defined by Weiss: the R^n-functional controllability [155][114] (see also in [129]), in which there is not this zero-input constraint. The definition for a single-delay system without delay on the input is as follows.

DEFINITION. *The linear system (10) is (ψ, R^n)-controllable (with regard to some function $\psi \in \mathcal{C}$) if, for any initial condition $\varphi \in \mathcal{C}$, there is a finite time $t_1 > 0$ and a control law $u(t) \in \mathcal{L}_2([0, t_1 + h], R^m)$ such that $x(t; \varphi, u) = \psi(t - t_1 - h)$ for all $t \in [t_1,\ t_1 + h]$.*

This property can be checked by generalizing the notion grammian [155] as in equation (38).

THEOREM 4.2. *The linear system (10) is $(0, R^n)$-controllable (i.e. with regard to $\psi = 0$) if*

1) *there is a finite time $t_1 > 0$ such that*

$$\text{rank}\left(\int_0^{t_1} F(t_1 - \theta) B_0 B_0^T F(t_1 - \theta)^T d\theta\right) = n, \tag{38}$$

with $F(t)$ solution of $\dot{F}(t) = A_0 F(t) + A_1 F(t - h)$, $F(0) = I$, $F(t_1) = 0$,

2) *the equation $A_0 x(t - h) + B_0 u(t) = 0,\ t \in [t_1, t_1 + h]$ has a solution $u(t) \in \mathcal{L}_2([t_1, t_1 + h], R^m)$.*

Condition 1) ensures R^n-point-wise controllability at time t_1, whereas condition 2) allows the solution to be maintained at the origin after t_1. Condition 1) can be replaced by simpler point-wise controllability conditions, which are recalled in the next subsection.

4.2. Spectral properties. The following spectral properties, as we shall see, constitute very interesting bases for effective control of linear systems. Spectral controllability can be seen as a functional controllability property, but it only applies to the problem of controlling the *spectrum* of the linear system (8) with model over ring (13),

$$\sigma(A) = \left\{s \in C, \quad \det\left(sI - \mathbf{A}(e^{-\delta s})\right)\right\}, \tag{39}$$

in such a way it belongs to some region of the complex, left half-plane. Of course, spectral properties concern the problem of stabilization (functional controllability to zero), but they have also been related to behavioral properties [125][124]. We will not consider here the infinite-dimension models, however, the spectral properties can also be tested within this framework (see [69]).

DEFINITIONS. *The system (8) or (13) is spectrally controllable if, for any $s \in C$,*

$$\text{rank}\left[sI - \mathbf{A}(e^{-\delta s}),\ \mathbf{B}(e^{-\delta s})\right] = n. \tag{40}$$

It is spectrally observable if, for any $s \in C$,

$$\text{rank}\left[sI - \mathbf{A}(e^{-\delta s})^T,\ \mathbf{C}^T(e^{-\delta s})\right] = n. \tag{41}$$

It is stabilizable if there exists a causal control law which makes it asymptotically stable. It is detectable if there exists a causal, asymptotic observer of the solution $x(t) \in R^n$.

THEOREM 4.3. *The system (8) is stabilizable if and only if (40) holds for any $s \in C$, $\mathrm{Re}(s) \geq 0$. It is detectable if and only if (41) holds for any $s \in C$, $\mathrm{Re}(s) \geq 0$.*

This result was proven in a constructive way in [10], whose work also studied the question of the realization:

THEOREM 4.4. *Any causal transfer matrix on $R(s, e^{-\delta s})$ admits a stabilizable and spectrally observable realization. It also admits a detectable and spectrally controllable realization.*

It was also noted that in relation to delay systems, the notion of *minimal realization* (in the sense of spectral controllability and spectral observability) does not always exist (the transfer $\frac{1+e^{-2s}}{s+e^{-s}\pi/2}$ was taken as example [88]).

4.3. Point-wise controllability properties. Many other works have been devoted to point-wise structural properties: one of the main ones is the so-called *euclidean-space controllability*, or *R^n-controllability*. This means, let us recall, defined for trajectories considered in the vector-space R^n. These works probably started with Kirillova, Churakova and Gabasov (1967), Buckalo (1968), Weiss (1970), Zmood (1974) (see [129]). All approaches use the same definition, but lead to different conditions (sometimes equivalent). We recall here the definition for linear systems with single delay (note it corresponds to the notion of *reachability*), but it can easily be extended to multiple delay systems with input delays, when the delays are commensurate.

DEFINITION. *The linear system (8) is R^n-controllable at time t_1 if, for any initial condition $\varphi \in \mathcal{C}$ and $x_1 \in R^n$, there is a time $t_1 > 0$ and a control law $u(t) \in \mathcal{L}_2([0, t_1], R^m)$ such that $x(t_1; \varphi, u) = x_1$.*

It is R^n-controllable if such a time t_1 exists .

It is strongly R^n-controllable if it is R^n-controllable at any time $t_1 > 0$.

If the above R^n-controllability property is restricted to $x_1 = 0$, then the system is R^n-controllable to the origin.

The evolution of the trajectory after t_1 is then not constrained by these definitions (the trajectory may not stay at x_1, contrary to the non-delayed case and to functional controllability). Two other differences with ODEs are to be noted: the time t_1, in general cannot be smaller than the delay δ (except in the rare case of *strong* controllability) and the R^n-controllability is not equivalent to the R^n-controllability to the origin (the difference corresponds to the so-called *point-wise completeness*, whose additional property makes the two definitions equivalent. Completeness can be checked by matrix-type N.S.C. due to Zwerkin (1971).

Many criteria give R^n-controllability conditions (for instance, eqn. (38) is a condition due to Weiss). The basic one uses the Kirillova-Churakova operators (9):

THEOREM 4.5. *The single-delay system (10) is R^n-controllable to the origin if* [14]

$$\begin{aligned} n = \mathrm{rank}[&P_0(0)B_0, P_1(0)B_0, P_1(1)B_0, \\ &P_2(0)B_0, P_2(1)B_0, P_2(2)B_0, ..., P_{n-1}(n-1)B_0]. \end{aligned}$$

4.4. Controllability over rings. The following, pointwise notions are of algebraic type and are detailed in [82][129][118]: roughly speaking, *strong controllability* implies the existence of a non-anticipative feedback control based on the past values of the solution, *i.e.* $x(t)$, $x(t-\delta)$, $x(t-2\delta)$,... that one can say to be of "polynomial

[14] *This condition is also necessary if the system is pointwise complete.*

type". *Weak controllability* just needs a "rational" feedback to exist, and the resulting control law may be anticipative (thus, non realizable). A link between this form of controllability and the subsystems description [114][142] is given in [118].

DEFINITION. *System over ring (13) is controllable over the ring* $R[\nabla]$ *or "strongly controllable", if there exists a control law of polynomial type* $u(t) = f(x, \nabla x, \nabla^2 x, ...)$, *allowing one to reach any element of the module* $R^n[\nabla]$ *from any initial state* $x_0 \in R^n[\nabla]$. *It is controllable over the field* $R(\nabla)$ *or "weakly controllable", if there exists a control law of rational type* $u(t) = f(x, \nabla x, \nabla^2 x, ..., \nabla^{-1} x, \nabla^{-2} x,)$ *allowing one to reach any element of the module* $R^n[\nabla]$ *from any initial state* $x_0 \in R^n[\nabla]$.

The following theorem (see a more complete version in [82]) uses the notations

$$\langle \mathbf{A}/\mathbf{B} \rangle = \left[\mathbf{B}, \mathbf{AB}, \mathbf{A}^2\mathbf{B}, ..., \mathbf{A}^{n-1}\mathbf{B}\right],$$

and $\langle \mathbf{A}/ \operatorname{Im} \mathbf{B} \rangle$ for the controllability submodule associated with the pair $(\mathbf{A}, \mathbf{B})$, *i.e.*

$$\langle \mathbf{A}/ \operatorname{Im} \mathbf{B} \rangle = \operatorname{Im} \mathbf{B} + \mathbf{A}^2 \operatorname{Im} \mathbf{B} + ... + \mathbf{A}^{n-1} \operatorname{Im} \mathbf{B}.$$

THEOREM 4.6. *The following, equivalent conditions are necessary and sufficient for system (13) to be strongly controllable –i.e. over* $R[\nabla]$*:*
1) $\langle \mathbf{A}(\nabla)/ \operatorname{Im} \mathbf{B}(\nabla) \rangle = R^n[\nabla]$;
2) *the Smith form of* $\langle \mathbf{A}(\nabla)/ \operatorname{Im} \mathbf{B}(\nabla) \rangle$ *is* $[I_{n\times n} \mid 0]$;
3) $\operatorname{rank}[sI - \mathbf{A}(z) \mid \mathbf{B}(z)] = n$ *for all* s *and* z *in* C.

THEOREM 4.7. *The following, equivalent conditions are necessary and sufficient for system (13) to be weakly controllable –i.e. over* $R(\nabla)$*:*
1) $\operatorname{rank}\langle \mathbf{A}(\nabla)/\mathbf{B}(\nabla) \rangle = n$;
2) *all the diagonal elements of the Smith form of* $\langle \mathbf{A}(\nabla)/ \operatorname{Im} \mathbf{B}(\nabla) \rangle$ *are nonzero*;
3) $\operatorname{rank}[sI - \mathbf{A}(z) \mid \mathbf{B}(z)] = n$ *for all* s *and at least one* z *in* C.

In the first statement, condition 2) means that $\langle \mathbf{A}(\nabla)/\mathbf{B}(\nabla) \rangle$ is $\mathcal{R}_u(\nabla)$-closed (see footnote 3). In the second statement, it is not.

In the same framework of models over rings, the notion of *controllability indices* has been extended to delay systems by Sename, Picard and Lafay [129][130][118], giving interesting information about the smallest time t_1 that is needed for given state variables (controllability submodules) to reach the expected value x_1 in R^n. This question is clearly illustrated by the very simple example [129] $\dot{x}_1(t) = u_1(t)$, $\dot{x}_2(t) = u_2(t-h)$ where different, minimal delay times are needed for the control of x_1 and x_2. Then, each of these indices are represented by a class and an order in this class: class reflects the minimal delay, whereas order corresponds to the classical notion for systems without delay (*i.e.*, the lengths of controllability chains). Controllability submodules are also associated (see details in [118]).

4.5. A summary of the controllability notions.

4.5.1. *Abridged notions.* Functional properties

- $\mathcal{M}_2$*-strict controllability*: $\exists\, u, \exists t_1 \to x_{t_1} = \varphi_1$;
- $\mathcal{M}_2$*-approximate controllability:* $\exists$ *series* $u_n, \exists t_1 \to \lim\, [x_{t_1}]_n = \varphi_1$;
- *absolute controllability* (linear, input delays only): $\exists\, u, u_{t_1} = 0$ and $x_{t_1} = \varphi_1$;
- R^n*-functional controllability* (linear, state delay only);
- *spectral controllability* (linear, commensurate): $\exists u, \sigma(A) = \{\lambda_i\}$.

Pointwise properties (for linear, commensurate delay systems)

- R^n*-controllability:* $\exists u, \exists t_1 \to x(t_1) = x_1$;
- *strong* R^n*-controllability:* $\exists u, \forall t \to x(t_1) = x_1$;
- R^n*-controllability to the origin:* $\exists u, \exists t_1 \to x(t_1) = 0$.

The question of the nature of control $u(t)$

- *(strong) controllability over the ring* $R[\nabla]$: polynomial $u(t) = \mathbf{K}(\nabla)x(t)$;
- *(weak) controllability over the field* $R(\nabla)$: rational $\mathbf{L}(\nabla)u(t) = \mathbf{K}(\nabla)x(t)$.

4.5.2. *Order relations between the controllabilities.* The survey paper [82] creates the following implications (and other additional ones, using the notion of torsion submodules):

THEOREM 4.8. *In the case of a linear system with commensurate delays (8), the following implications hold:*

1) *Strong controllability, over* $R[\nabla] \Rightarrow$ *Absolute controllability* $\Rightarrow$ *Weak controllability, over* $R(\nabla) \Rightarrow R^n$*-controllability.*

2) *Approximate controllability* $\Rightarrow$ *Spectral controllability* $\Rightarrow$ *Weak controllability, over* $R(\nabla)$.

Note that it allows one to conclude that strong controllability is a very demanding property: in fact, it means that the system can be controlled as if it were not including any delay.

4.6. Remarks on the observability. The previous notions of controllability (strong, weak, spectral,...) can be transposed to observability (see [118][121] and references therein). Indexes and classes can be used for determining the minimum time needed by an observer to construct the point $x(t)$. General solution can be obtained for retarded systems by means of realizations over $\mathcal{R}_u(\nabla)$ but, as far as neutral systems are concerned, this problem of reconstruction is still open [118].

In the neutral case, the main problem is that, when trying to reconstruct the instantaneous value of $x(t)$, there is no asymptotic cancellation of the initial gap function (difference between the system / plant and the observer / computer): this yields a bias on the estimated vector $x(t)$, except in the particular case of *formally stable* systems[15]. Besides, for any equation $z(t) = n(\nabla)y(t)$, where $n(\nabla)$ is a rational fraction defined over $\mathcal{R}_u(\nabla)$, function $z(t)$ is, in general, only defined *modulo* an initial function φ on z. These (connected) reasons means that realization of asymptotic observers (in open or closed loop) is "only" solved for formally stable systems.

5. Control. Since the Smith "posicast control" [133] and predictor [134][135], control of delay systems has been widely considered. A great part of the practices were based on approximation methods, which are not necessarily convenient when significant uncertainties -including delay variations- are involved in the process. We have previously mentioned some approaches in the subsection "Stabilization" but, of course, the quasi-totality of the control methods received attempts at a generalization. This section will provide a glance at some present trends.

5.1. Spectrum assignment. In the 70s, some papers emphasized the interest of using distributed-delays controllers for discrete-delays plants [8][104][66][99][98] (see also [152] [154][10]): such operators, placed in the feedback loop, allow a reduction of the spectrum $\sigma(A)$ (39) to a finite set. Contrarily to the problem (initiated by Osipov in 1965, see [98]) of shifting an arbitrary but finite number of eigenvalues, finite-spectrum assignment does not require the preliminary knowledge of the spectrum $\sigma(A)$; moreover, stability of the closed loop is easy to check, since the characteristic function (22) becomes a polynomial. The following simple, scalar example [10] can

[15] As defined in Section 3.12, a linear system with rational fraction $n(\nabla)$ over $\mathcal{R}_u(\nabla)$ is *formally stable* if its denominator is asymptotically stable (in Schur-Cohn sense).

illustrate the idea:

$$(42) \qquad \dot{y}(t) = y(t) + u(t-1), \qquad \frac{Y(s)}{U(s)} = \frac{e^{-s}}{s-1},$$
$$u(t) = -2\int_0^1 e^{\theta} u(t-\theta)d\theta - 2ey(t) + v(t).$$

Here, from (17), the control $u(t)$ achieves a finite-spectrum assignment at $s = -1$ (note that the Laplace transform of the control $u(t)$ (42) is $\mathcal{L}(u(t)) = U(s) = V(s) - \frac{2e(1-s)}{-1-s+2e^{1-s}}Y(s)$. A simulation result is given Figure 8: here, the distributed effect was approximated by a discrete one:

$$\int_0^1 e^{\theta}u(t-\theta)d\theta \simeq \frac{1}{12}\Big[u(t) + 4e^{\frac{1}{4}}u(t-\frac{1}{4}) + 2e^{\frac{1}{2}}u(t-\frac{1}{2}) + 4e^{\frac{3}{4}}u(t-\frac{3}{4}) + e^{1}u(t-1)\Big].$$

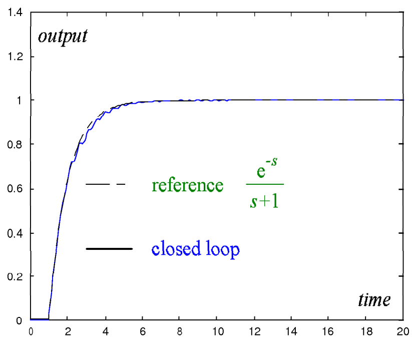

Figure 8: finite pole assignment (42).

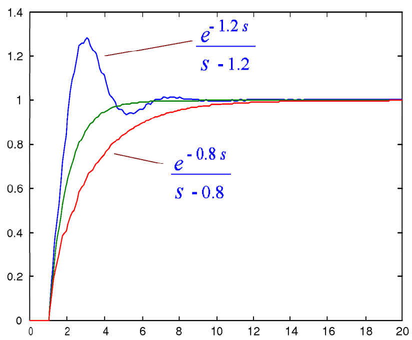

Figure 9: robustness test.

The following result was proven (necessity [98], sufficiency [153]):

THEOREM 5.1. *The system (8) is finite-spectrum assignable if and only if it is spectrally controllable.*

Several algorithms followed, proposing calculation of the corresponding feedback in the general case. A complete algebraic formalism was recently proposed [10][11][91], based on the set of pseudo-polynomials $\mathcal{E}$ (see previous section “Algebraic formalism”). The sketch of solution is as follows: if the expected finite spectrum is defined by the polynomial equation $\phi(s) = 0$, $\phi \in R[s]$, if the process is described by $Y(s)/U(s) = p(s)/q(s)$, $p, q \in R[s, e^{-s}]$, and if the control is to be calculated as $U(s) = p_c(s)/q_c(s)$, $p_c, q_c \in \mathcal{E}$, then the problem has a solution if $qq_c + pp_c = \phi$ has a solution. This last condition holds because $\mathcal{E}$ is a Bézout domain. On these bases, the robustness aspects now remain to be studied: for instance, Figure 9 shows that for example 42, the result remains interesting under *parameter variations* $\pm 20\%$. Besides, robustness in relation to the *sampling periods* (time scaling of simulation, of output sampling and/or of realization of the distributed effect by means of discrete delays) is also a problem to be taken into consideration (changing the sampling period in simulations of Figures 8-9 can destabilize the resulting behaviors. Other problems concern the practical implementation of this kind of control: in [144], it is shown that an accurate approximation of integral terms may lead to the instability of the closed-loop when delay occurs in the input.

5.2. Some other control aspects. Many control problems can be studied by means of models over rings, then for the synthesis of discrete-delayed feedback laws: *disturbance decoupling* [18] and *block-decoupling* [19], *model matching* [120], *pre-compensators design* [119]. Some overview and results can be found in [118].

Concerning *optimal control*, many results and references can be found in [80] (also considering stochastic FDEs). [44][30] considered some approaches by approximation of infinite-dimensional Riccati equations, and additional results and references on LQG control are given in [116].

Self-adjusting control with reference model (with identification problem) is considered in [80]. H_∞*-robustness* results and references can be found in [145][38][154][109]. *Feedback linearization* of delay systems was considered in [43], and *vibrational control* in [87] [33]. *Constrained control* (with invariant or saturated control) was considered in [26][110][33][52][62] [3], and *deadbeat control* of ODEs by means of delays can be found in [154]. Finally, one can mention some trends in *crone control* (french abbreviation of "robust control with non-integer order of derivation") [65] and *sliding mode control* [94][39][24][54][53][126].

Specific control for nonlinear systems may be found in [146][147], [2]. Note that in this last paper, the control objective is reached through the limitation of the amplitude of a limit cycle which is done by adding a delay term in the control. The main tool used in this approach is the study of Hopf bifurcation [61].

Of course, if control can be studied with some success in the case of unperturbed, linear, time-invariant models, it is clear that, in more complex cases, the domain remains wide open.

6. Conclusion. This overview of three aspects of delay systems -modelling, stability and controllability- makes appear four points of view:

1- the *functional* point of view: FDEs and infinite-dimensional models, Lyapunov-Krasovskii functionals for stability, functional and spectral controllability properties;

2- the *pointwise* point of view: models over rings, Lyapunov-Razumikhin functions for stability, pointwise R^n-controllability properties;

3- the *approximative* one: mainly based on classical, finite-dimension simplifications, followed by usual criteria for ordinary differential systems.

Only the first two approaches take into consideration the specific characters of delay systems. Roughly speaking, the first class is the only one that allows nonlinear behaviors to be considered.

In the author's opinion, several interesting control methods can now to be applied on concrete processes for which a linear model with constant, commensurate delays is available. In the contrary case (nonlinear models, time-varying delays), the "toolbox" is still reduced to stabilization results (with, as usual, some conservative properties).

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