Stability of perturbed systems with time-varying delays

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Abstract

This paper gives easily verifiable sufficient conditions of robust asymptotic stability of linear time-delay systems subject to parametric unstructured or highly-structured perturbations. The criteria given in this paper are delay-independent or delay-dependent. The considered delay may be time-varying. An estimation of the transient behaviour of the studied systems is also provided (exponential rate of convergence).

Scalar or vectorial inequalities involving Hurwitz matrices, matrix measures and norms constitute the mathematical foundations of the exposed results.

Keywords: Time-varying delay; Perturbations; Robust asymptotic stability; Comparison principle; M-matrix; Matrix measure

1. Introduction

The stability of time-delay systems has been studied intensively during the past decades. In fact, time-delays, due to transportation lags, finite calculation times, measurement times, etc., appear in numerous industrial and natural processes, often leading to oscillations and sometimes instability. It is therefore essential to study their effects on the responses of the systems, particularly when considering closed-loop control.

Numerous works deal with the stability of time-delay systems [5, 8, 10]. Some criteria are directly obtained from the characteristic equation [15, 19, 23, 24, 18], sometimes involving the determination of eigenvalues or norms of matrices [1, 13, 20]. Others involve the Lyapunov–Razumikhin theorem and Riccati or Lyapunov equations [8, 10, 16]. Others deal with scalar conditions in terms of matrix measures and norms [11, 14, 25], or matrix ones in terms of Hurwitz matrices [3, 7, 22].

The conditions given in this paper are of the last two types.

When studying the stability of an industrial process, it is almost always necessary to use a criterion which is delay-dependent. Moreover, the stability property of a working point is really robust if it still holds when perturbations make the model vary. In practice, the perturbed parameters include the delay. In spite of these remarks, only a few papers [7, 17] give delay-dependent theorems valid for time-varying delays among those

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dealing with the stability of perturbed time-delay systems. Our aim is to improve these criteria and to break away from some of the hypotheses of previous articles.

The systems which are in our interest can be written in the following form:

$$\dot{x}(t) = Ax(t) + Bx(t - \tau(t)) + f(x(t), t) + g(x(t - \tau(t)), t) \quad \text{for } t \geqslant 0 \text{ and } x(t) \in \mathbb{R}^n, \tag{1}$$

$x_0(\theta) = \varphi(\theta)$ for $\theta \in \{\lambda \in \mathbb{R},\ \lambda \leqslant 0 : \exists a \geqslant 0/\lambda = a - \tau(a)\}$, where φ is piecewise continuous.

The functions f and g are unknown and represent the perturbations, structured or not [11]. They satisfy $f(0, t) = 0$ and $g(0, t) = 0$ whatever $t \geqslant 0$. The delay $\tau(t)$ is piecewise continuous and verifies $0 \leqslant \tau(t) \leqslant \tau_m$. The existence and uniqueness of the solution of the problem (1) are assumed.

Our paper is organized as follows. The next section is concerned with the scalar criterion. The perturbations considered are unstructured. A condition of asymptotic stability is given, as well as a way to determine an estimation of the transient response of the system. This criterion is compared with other criteria. Section 3 deals with a matrix criterion expressed in terms of M-matrices. The perturbations are this time highly structured. The last section deals with an example of application of the different theorems.

Notations. The capital letters without indices represent matrices; the small ones represent either vectors or scalars. $|\cdot|$ denotes the absolute value of a real.

If x is a vector, $|x|$ denotes the vector whose components are the absolute values of the components of x. $\mathrm{Sup}_{0 \leqslant \lambda \leqslant a}|x(t - \lambda)|$ is the vector whose ith entry is $\mathrm{Sup}_{0 \leqslant \lambda \leqslant a}|x_i(t - \lambda)|$.

If $A = (A_{ij})$, then $|A| = (|A_{ij}|)$, and $A^* = (A^*_{ij})$ with $A^*_{ii} = A_{ii}$ and $A^*_{ij} = |A_{ij}|$ for $i \neq j$. $\|\cdot\|$ denotes any norm in $\mathbb{R}^n$ or its induced matrix norm. $\mu(\cdot)$ is the associated matrix measure, defined by

$$\mu(A) = \lim_{h \to 0^+} \frac{1}{h}[(\|I + hA\| - 1)] \qquad [4].$$

A matrix is said to be negative (respectively positive) if each of its elements is negative (resp. positive). The vector inequality $v \leqslant w$ has to be understood as n inequalities $v_i \leqslant w_i$.

A matrix A is said to be an M-matrix if its off-diagonal elements are nonpositive and if the real parts of its eigenvalues are positive. If A is such a matrix, $\lambda(A)$ denotes the minimum real part of its eigenvalues. It can be proved that $\lambda(A)$ is positive and is an eigenvalue of A. Moreover, if A is irreducible, an eigenvector associated with this eigenvalue $\lambda(A)$ may be chosen with positive components. Several definitions, and properties concerning M-matrices are given in [6, 9].

If χ is a real, $\mathrm{sgn}(\chi) = 1$ if $\chi \geqslant 0$; $\mathrm{sgn}(\chi) = -1$ if $\chi < 0$.

2. Unstructured perturbations (scalar criterion)

Let us decompose the matrix B in the following way: $B = B' + B''$. The system (1) is considered with unstructured perturbations:

$$\|f(x, t)\| \leqslant \alpha\|x\|, \qquad \|g(x, t)\| \leqslant \beta\|x\|.$$

Theorem 1. *If*

$$\mu(A + B') + \alpha + \beta + \|B''\| + \tau_m(\|B'A\| + \|B'B\| + \alpha\|B'\| + \beta\|B'\|) < 0,$$

then the equilibrium 0 *of the system* (1) *is asymptotically stable. Furthermore, the following inequality:* $\|x(t)\| \leqslant \mathrm{Sup}_{-\tau_m \leqslant \theta \leqslant \tau_m}\{\|x(\theta)\| \cdot \mathrm{e}^{\sigma\theta}\}\mathrm{e}^{-\sigma t}$ *holds, whenever* $t \geqslant \tau_m$, *where* σ *is the unique positive real solution of the following equation*:

$$\sigma + (\tau_m(\|B'A\| + \alpha\|B'\|) + \beta + \|B''\|)\mathrm{e}^{\sigma\tau_m} + \tau_m(\|B'B\| + \beta\|B'\|)\mathrm{e}^{2\sigma\tau_m} + \mu(A + B') + \alpha = 0. \tag{2}$$

Proof. see Appendix A. □

Remark. (1) One key point of this criterion is the decomposition of the matrix B into $B = B' + B''$, where B' is chosen such that $A + B'$ is "more stable" than A. Roughly, this decomposition corresponds to a decomposition of the delayed terms into two groups: the stabilizing ones and the destabilizing ones. This technique enables one to take the stabilizing effect of part of the delayed terms into account. The example given at the end of this paper compares the results obtained by setting $B' = B$, $B'' = 0$ (no decomposition), with the results obtained with a decomposition of B. Let us remark that a delay-independent criterion is obtained if B' is set to 0.

(2) A partial optimization of the decomposition is possible, but is not necessarily interesting: the optimization is quite long, and moreover "natural" decompositions which lead to very interesting results are easily found (see the first example). These natural decompositions are chosen such that $A + B'$ is sufficiently stable and $\|B''\| + \tau_m(\|B'A\| + \|B'B\| + \alpha\|B'\| + \beta\|B'\|)$ is not too big.

(3) As stated in the introduction, this criterion can be compared to several results found in the literature [25, 17]. In addition to the preceding remark (1), comparisons may be made with other results: Wang et al. [25] considered delay-dependent as well as delay-independent criteria, but did not allow the delay(s) to be time-varying. Moreover, as $\|CD\| \leqslant \|C\|\ \|D\|$ whatever the matrices C and D, our criterion is more precise. Niculescu et al. [17] considered time-varying delays, and both delay-dependent and delay-independent criteria. Theorem 1 is similar to the delay-dependent criterion they gave, but the domain of application of the theorem proposed here is wider (no restriction on the derivative $\dot{\tau}(t)$, no need to know $\tau(0)$ for the overvaluation of $\|x(t)\|$).

3. Highly structured perturbations (matrix criterion)

The perturbations on the system (1) are now highly structured

$$|f(x, t)| \leqslant F|x|, \qquad |g(x, t)| \leqslant G|x|, \quad \text{where } F > 0 \text{ and } G > 0.$$

Theorem 2. *If the real parts of the eigenvalues of the matrix* $M = (A + B')^* + |B''| + F + G + \tau_m(|B'A| + |B'B| + |B'|(F + G))$ *are negative (or equivalently M is the opposite of an M-matrix), then the equilibrium* 0 *of the system* (1) *is asymptotically stable.*

Moreover, if M is irreducible, then $|x(t)| \leqslant k\cdot \mathrm{e}^{-\gamma t}$ *for* $t \geqslant \tau_m$, *as soon as* $|x(\theta)| \leqslant k \cdot \mathrm{e}^{-\gamma t}$, $\theta \in [-\tau_m, \tau_m]$, *where k and γ are determined as follows*:

- $P_\sigma = -(A + B')^* - F - [\tau_m(|B'A| + |B'|F) + |B''| + G]\mathrm{e}^{\sigma\tau_m} - \tau_m(|B'B| + |B'|G)\mathrm{e}^{2\sigma\tau_m}$.
- *γ is the real positive solution of the equation* $\lambda(P_\gamma) = \gamma$ *(see the notations for the definition of* $\lambda(P_\gamma)$*).*
- *k is a positive eigenvector of* P_γ *associated with the eigenvalue γ.*

Proof. See Appendix B. The proofs of Theorems 1 and 2 are based on Lemma 1 (see Appendix C), which is a generalization of a theorem given in [22]. □

Remark. The matrix criterion is more precise than the scalar criterion, because of the use of a comparison system with the same dimension as the initial one. Moreover, the perturbations are modelled more accurately.

It generalizes the delay-independent results published by Dambrine et al. [3]. It also improves the criteria given in Goubet et al. [7]: in fact, as the matrix $|CD|$ is always smaller than the matrix $|C|\cdot|D|$, the stability criterion written here is more interesting. Moreover, this article provides estimates of the asymptotic rate of convergence, which are not given in [7], and allows the delay and the initial function to be piecewise continuous.

Theorem 2 enables the proof of the *robust* stability: the considered delay is time-varying and not necessarily known. As for the uncertainties or perturbing terms, the bounds only are necessary for the proof of the stability. Of course, the same remark could have been written after Theorem 1.

4. Examples

Two examples are considered in this section: the first one aims at comparing our results with the theorems of the same kind [25, 17, 7] (formal comparisons have already been made below the two theorems). The second one has already been studied in [16, 12] with other methods.

Example 1. Let us consider the following system:

$$\frac{\mathrm{d}x}{\mathrm{d}t}(t) = Ax(t) + Bx(t-\tau(t)) + Gx(t-\tau(t)),$$

where

$$A = \begin{bmatrix} -1.2 & 0.1 \\ -0.1 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} -0.6 & 0.7 \\ -1 & -0.8 \end{bmatrix}, \quad Gx(t-\tau(t)) = \begin{bmatrix} \beta(x(t-\tau(t)),t))x_1(t-\tau(t)) \\ 0 \end{bmatrix},$$

with $|\beta(x(t-\tau(t)), t)| \leqslant 0.1$ whatever the values of the parameters. τ, the initial function, and β are piecewise continuous.

Our criteria, as well as the ones proved by other authors, have been applied to this system. The norm used is the following one: $\|y\| = |y_1| + |y_2|$. The associated matrix norm and measure are

$$\|M\| = \mathrm{Max}\{|M_{11}| + |M_{21}|; |M_{12}| + |M_{22}|\}; \quad \mu(M) = \mathrm{Max}\{M_{11} + |M_{21}|; |M_{12}| + M_{22}\}.$$

The results obtained with the different theorems are given in Table 1 (the rates of convergence are given for $\tau_m = 0.1$).

Let us remark that the last decomposition has been obtained with an optimization procedure which is not detailed here. The example shows that the "natural" decomposition

$$B = \begin{bmatrix} -0.6 & 0 \\ 0 & -0.8 \end{bmatrix} + \begin{bmatrix} 0 & 0.7 \\ -1 & 0 \end{bmatrix}$$

leads to very interesting results, and that an optimization is not necessary.

Example 2. The following system has been studied in [16, 12]:

$$\dot{x}(t) = Ax(t) + Bx(t-\tau(t)) + f(x(t), t) + g(x(t-\tau(t)), t)$$

with

$$A = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix}, \qquad f(x(t), t) = \begin{bmatrix} \alpha_1 \cos(t) & 0 \\ 0 & \alpha_2 \sin(t) \end{bmatrix},$$

$$B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad g(x(t-\tau(t)), t) = \begin{bmatrix} \beta_1 \cos(t) & 0 \\ \varepsilon & \beta_2 \cos(t) \end{bmatrix}.$$

The upper bounds of the uncertainties are: $|\alpha_1| \leqslant 1.6$; $|\beta_1| \leqslant 0.1$; $|\alpha_2| \leqslant 0.05$; $|\beta_2| \leqslant 0.3$. The results of the two publications are the following: $\tau_m < 0.1036$ [16], $\tau_m < 0.2013$ [12]. Let us apply the theorems of this article without any decomposition: $B' = B$, $B'' = 0$.

The system can be shown to be asymptotically stable for any piecewise continuous delay $\tau(t) < 0.276$. This result is better than the ones given before for the same system. Moreover, the hypotheses on the delay are less strong. We could even have considered elements of more general forms, for example a nonlinear parameter α_1 such that $|\alpha_1(t, x(t), x(t-\tau(t)))| \leqslant 1.6$ (instead of $\alpha_1 \cos(t)$).

Remark. It is known that sufficient and necessary conditions of stability cannot be found for such uncertain nonlinear systems. Our results reveal to be less restrictive than the previous ones, especially the matrix

Table 1

Criteria	Hypotheses	Results
Wang et al. [25]	Continuous initial function $\tau(t)$ constant	Stability if $\tau_m < 0.125$, $\|x(t)\| \leqslant \mathrm{Sup}_{-\tau_m \leqslant s \leqslant \tau_m}\{\|x(s)\|\} e^{-0.11t}$
Niculescu et al. [17]	The initial function, τ and β are continuous τ is differentiable $0 \leqslant \dot{\tau}(t) \leqslant \alpha < 1$	Stability for $\tau_m < 0.157$ Exponential rate of convergence $\sigma = 0.20$ if $\alpha = 0$ ($\tau(t) = \tau_m$), $\sigma = 0.10$ if $\alpha = 0.5$ and $\tau(0) = 0.05$
Goubet et al. [7]	The initial function, τ and β are continuous	Stability for $\tau_m < 0.382$ with the decomposition $B = \begin{bmatrix} -0.6 & -0.06 \\ 0.067 & -0.8 \end{bmatrix} + \begin{bmatrix} 0 & 0.76 \\ -1.067 & 0 \end{bmatrix}$
Criteria of this article	The initial function, τ and β, are piecewise continuous	Scalar criterion: *without decomposition: $\tau_m < 0.157$, *with the splitting-up $B = \begin{bmatrix} -0.6 & 0 \\ 0 & -0.8 \end{bmatrix} + \begin{bmatrix} 0 & 0.7 \\ -1 & 0 \end{bmatrix} : \tau_m < 0.286$, $\|x(t)\| \leqslant \mathrm{Sup}_{-\tau_m \leqslant \theta \leqslant \tau_m}\{\|x(\theta)\| \cdot e^{0.340}\} e^{-0.34t}$ Matrix criterion: *without decomposition: $\tau_m < 0.260$, *with the previous decomposition: $\tau_m < 0.417$, $\lvert x(t)\rvert \leqslant \alpha \begin{bmatrix} 1 \\ 1.1 \end{bmatrix} \cdot e^{-0.55t}$, with α such that $\lvert x(\theta)\rvert \leqslant \alpha \begin{bmatrix} 1 \\ 1.1 \end{bmatrix} \cdot e^{-0.55\theta}$, $\theta \in [-0.1, 0.1]$, *with the decomposition $B = \begin{bmatrix} -0.6 & -0.06 \\ 0.067 & -0.8 \end{bmatrix} + \begin{bmatrix} 0 & 0.76 \\ -1.067 & 0 \end{bmatrix} : \tau_m < 0.429$

conditions that are less restrictive than the scalar ones. This has been shown on the different examples that have been studied.

Conclusions

The delay-dependent criteria given in this paper enable the study of processes with a piecewise-continuous time-varying delay without any knowledge of this delay except its upper bound; the criteria are easily checkable. They also allow for the estimation of the transient responses of the models, and can easily be generalized to systems with several delays. The stability is considered with regard to unstructured or highly structured perturbations, as classically defined in the robustness studies.

The results are extensions of several articles listed in this paper. One finding is the decomposition of the delayed matrix.

Appendix A

Proof of Theorem 1. The equation of the system can be rewritten in the following way: whenever $t \geqslant \tau_m$

$$\dot{x}(t) = (A + B')x(t) - B' \int_{t-\tau(t)}^{t} \dot{x}(s)\,\mathrm{d}s + B''x(t - \tau(t)) + f(x(t), t) + g(x(t - \tau(t)), t).$$

Replacing $\dot{x}(s)$ by its value $Ax(s) + Bx(s - \tau(s)) + f(x(s), s) + g(x(s - \tau(s)), s)$, it yields

$$\dot{x}(t) = (A + B')x(t) - B'A \int_{t-\tau(t)}^{t} x(s)\,ds - B'B \int_{t-\tau(t)}^{t} x(s - \tau(s))\,ds$$
$$- B' \int_{t-\tau(t)}^{t} f(x(s), s)\,ds - B' \int_{t-\tau(t)}^{t} g(x(s - \tau(s)), s)\,ds$$
$$+ B''x(t - \tau(t)) + f(x(t), t) + g(x(t - \tau(t)), t). \tag{A.1}$$

Let us evaluate the rate of change of the norm of $x(t)$: (right-hand derivative)

$$\frac{d^+ \|x(t)\|}{dt} = \lim_{h \to 0^+} \frac{\|x(t + h)\| - \|x(t)\|}{h} = \lim_{h \to 0^+} \frac{1}{h}[\|x(t) + h\dot{x}(t)\| - \|x(t)\|].$$

Using the properties of the associated norms of vectors and matrices, the following inequality is obtained:

$$\frac{d^+ \|x(t)\|}{dt} \leqslant \lim_{h \to 0^+} \frac{1}{h}[\|x(t) + h(A + B')x(t)\| - \|x(t)\|] + \|B'A\| \int_{t-\tau(t)}^{t} \|x(s)\|\,ds$$
$$+ \|B'B\| \int_{t-\tau(t)}^{t} \|x(s - \tau(s))\|\,ds + \|B'\| \int_{t-\tau(t)}^{t} \|f(x(s), s)\|\,ds$$
$$+ \|B'\| \int_{t-\tau(t)}^{t} \|g(x(s - \tau(s)), s)\|\,ds + \|B''\|\,\|x(t - \tau(t))\| + \|f(x(t), t)\| + \|g(x(t - \tau(t)), t)\|$$
$$\leqslant \left\{ \lim_{h \to 0^+} \frac{1}{h}[(\|I + h(A + B')\| - 1)] + \alpha \right\} \|x(t)\|$$
$$+ [\tau_m(\|B'A\| + \|B'\|\alpha) + \|B''\| + \beta] \sup_{0 \leqslant \lambda \leqslant \tau_m} \|x(t - \lambda)\|$$
$$+ \tau_m(\|B'B\| + \|B'\|\beta) \sup_{0 \leqslant \lambda \leqslant 2\tau_m} \|x(t - \lambda)\|, \quad \text{where } I \text{ is the identity matrix.}$$

As

$$\lim_{h \to 0^+} \frac{1}{h}[(\|I + h(A + B')\| - 1)] = \mu(A + B') \quad \text{(see [4])},$$

$$\frac{d^+ \|x(t)\|}{dt} \leqslant [\mu(A + B') + \alpha]\,\|x(t)\| + [\tau_m(\|B'A\| + \|B'\|\alpha) + \|B''\| + \beta] \sup_{0 \leqslant \lambda \leqslant \tau_m} \|x(t - \lambda)\|$$
$$+ \tau_m(\|B'B\| + \|B'\|\beta) \sup_{0 \leqslant \lambda \leqslant 2\tau_m} \|x(t - \lambda)\|.$$

Then the criterion immediately follows, using Lemma 1 (see Appendix C). As a solution of the differential equation

$$\frac{d^+ z(t)}{dt} = [\mu(A + B') + \alpha]\,z(t) + [\tau_m(\|B'A\| + \|B'\|\alpha) + \|B''\| + \beta] \sup_{0 \leqslant \lambda \leqslant \tau_m} z(t - \lambda)$$
$$+ \tau_m(\|B'B\| + \|B'\|\beta) \sup_{0 \leqslant \lambda \leqslant 2\tau_m} z(t - \lambda)$$

is $a\mathrm{e}^{-\sigma t}$, where σ is given by (2), $\|x(t)\| \leqslant a\mathrm{e}^{-\sigma t}$ if the same inequality is valid for $-\tau_m \leqslant t \leqslant \tau_m$. □

Appendix B

Proof of Theorem 2. The inequality (A.1) of Appendix A leads to

$$\begin{aligned}\frac{\mathrm{d}^+|x_i|}{\mathrm{d}t}(t) = {} & \sum_{k=1}^{n}(A+B')_{ik}x_k(t)\,\mathrm{sgn}(x_i(t)) - \sum_{k=1}^{n}(B'A)_{ik}\int_{t-\tau(t)}^{t}x_k(s)\,\mathrm{d}s\,\mathrm{sgn}(x_i(t)) \\ & - \sum_{k=1}^{n}(B'B)_{ik}\int_{t-\tau(t)}^{t}x_k(s-\tau(s))\,\mathrm{d}s\,\mathrm{sgn}(x_i(t)) \\ & - \sum_{k=1}^{n}(B')_{ik}\int_{t-\tau(t)}^{t}f_k(x(s),s)\,\mathrm{d}s\,\mathrm{sgn}(x_i(t)) - \sum_{k=1}^{n}(B')_{ik}\int_{t-\tau(t)}^{t}g_k(x(s-\tau(s)),s)\,\mathrm{d}s\,\mathrm{sgn}(x_i(t)) \\ & + \sum_{k=1}^{n}(B'')_{ik}x_k(t-\tau(t))\,\mathrm{sgn}(x_i(t)) + f_i(x(t),t)\,\mathrm{sgn}(x_i(t)) + g_i(x(t-\tau(t)),t)\,\mathrm{sgn}(x_i(t)).\end{aligned}$$

As $x_i(t)\,\mathrm{sgn}(x_i(t)) = |x_i(t)|$ and as $|\mathrm{sgn}(x_i(t))| = 1$, an upperbound of $(\mathrm{d}^+|x_i|/\mathrm{d}t)(t)$ is easily calculated (see the valuations done in [3, 7] for more details):

$$\begin{aligned}\frac{\mathrm{d}^+|x_i|}{\mathrm{d}t}(t) \leqslant {} & \sum_{k=1}^{n}(A+B')^*_{ik}|x_k(t)| + \tau_m\sum_{k=1}^{n}(|B'A|)_{ik}\sup_{0\leqslant\lambda\leqslant\tau_m}|x_k(t-\lambda)| \\ & + \tau_m\sum_{k=1}^{n}(|B'B|)_{ik}\sup_{0\leqslant\lambda\leqslant 2\tau_m}|x_k(t-\lambda)| + \tau_m\sum_{k=1}^{n}(|B'|F)_{ik}\sup_{0\leqslant\lambda\leqslant\tau_m}|x_k(t-\lambda)| \\ & + \tau_m\sum_{k=1}^{n}(|B'|G)_{ik}\sup_{0\leqslant\lambda\leqslant 2\tau_m}|x_k(t-\lambda)| + \sum_{k=1}^{n}(|B''|)_{ik}\sup_{0\leqslant\lambda\leqslant\tau_m}|x_k(t-\lambda)| \\ & + \sum_{k=1}^{n}(F)_{ik}|x_k(t)| + \sum_{k=1}^{n}(G)_{ik}\sup_{0\leqslant\lambda\leqslant\tau_m}|x_k(t-\lambda)|.\end{aligned}$$

So

$$\begin{aligned}\frac{\mathrm{d}^+|x|}{\mathrm{d}t}(t) \leqslant {} & [(A+B')^* + F]|x(t)| \\ & + (\tau_m(|B'A| + |B'|\cdot F) + |B''| + G)\sup_{0\leqslant\lambda\leqslant\tau_m}|x(t-\lambda)| + \tau_m(|B'B| + |B'|G)\sup_{0\leqslant\lambda\leqslant 2\tau_m}|x(t-\lambda)|.\end{aligned}$$

As the off-diagonal elements of $(A+B')^* + F$ and all elements of $\tau_m(|B'A| + |B'|\cdot F) + |B''| + G$ and of $\tau_m(|B'B| + |B'|G)$ are nonnegative, Lemma 1 (see Appendix C) can be used; the theorem is proved. □

Appendix C. Comparison principle

The following lemma is a generalization of a comparison principle given in [22] to systems with two delays:

Lemma C1. *Let C, D, E be $n\times n$ matrices with real elements and let $x(t)$ be a solution of the differential inequality*

$$\dot{x}(t) \leqslant -Cx(t) + D\sup_{0\leqslant\lambda\leqslant\tau_1}x(t-\lambda) + E\sup_{0\leqslant\lambda\leqslant\tau_2}x(t-\lambda) \quad \textit{for } t\geqslant 0. \tag{C.1}$$

If $D \geqslant 0$, $E \geqslant 0$, if the off-diagonal elements of C are nonpositive, and if $(C - D - E)$ is an M-matrix, then a solution $x(t)$ of this inequality is overvalued by the asymptotically stable solution $z(t)$ of the differential equation

$$\dot{z}(t) = -Cz(t) + D \operatorname*{Sup}_{0 \leqslant \lambda \leqslant \tau_1} z(t-\lambda) + E \operatorname*{Sup}_{0 \leqslant \lambda \leqslant \tau_2} z(t-\lambda) \quad \text{for } t \geqslant 0 \tag{C.2}$$

as soon as $x(\theta) \leqslant z(\theta)$ for $-\mathrm{Max}(\tau_1, \tau_2) \leqslant \theta \leqslant 0$.

Moreover, if $(C - D - E)$ is an irreducible M-matrix, then there exist a constant $\gamma > 0$ and a constant vector $k > 0$ such that $x(t) \leqslant k\mathrm{e}^{-\gamma t}$, for $t \geqslant 0$. k and γ are obtained in the following way:

- *γ is the positive real solution of the equation $\lambda(A_\gamma) = \gamma$, where $A_\gamma = C - D\mathrm{e}^{\gamma\tau_1} - E\mathrm{e}^{\gamma\tau_2}$.*
- *k is a positive eigenvector of A_γ associated with the eigenvalue γ.*

Proof. (1) (*the scheme of the proof is the same as in* [22]): Let us consider the following inequality:

$$\dot{y}(t) > -Cy(t) + D \operatorname*{Sup}_{0 \leqslant \lambda \leqslant \tau_1} y(t-\lambda) + E \operatorname*{Sup}_{0 \leqslant \lambda \leqslant \tau_2} y(t-\lambda)$$

$$\text{with } y(\theta) > x(\theta), \quad -\mathrm{Max}(\tau_1, \tau_2) \leqslant \theta \leqslant 0.$$

Suppose there exist a time t and an index i such that $x_i(t) = y_i(t)$. Let us define $t_1 = \inf\{t : x_i(t) = y_i(t) \text{ for some } i\}$. Then $t_1 > 0$ and there exists an integer j such that $x_j(t_1) = y_j(t_1)$, and $x(t) < y(t)$ if $t < t_1$. The following inequalities hold:

$$\dot{x}_j(t_1) \leqslant -\sum_{k=1}^{n} C_{jk} x_k(t_1) + \sum_{k=1}^{n} D_{jk} \operatorname*{Sup}_{0 \leqslant \lambda \leqslant \tau_1} x_k(t_1-\lambda) + \sum_{k=1}^{n} E_{jk} \operatorname*{Sup}_{0 \leqslant \lambda \leqslant \tau_2} x_k(t_1-\lambda)$$

$$\leqslant -\sum_{k=1}^{n} C_{jk} y_k(t_1) + \sum_{k=1}^{n} D_{jk} \operatorname*{Sup}_{0 \leqslant \lambda \leqslant \tau_1} y_k(t_1-\lambda) + \sum_{k=1}^{n} E_{jk} \operatorname*{Sup}_{0 \leqslant \lambda \leqslant \tau_2} y_k(t_1-\lambda) < \dot{y}_j(t_1).$$

This is in contradiction with the fact that $x(t) < y(t)$ if $t < t_1$ and $x_j(t_1) = y_j(t_1)$. Thus $y(t) > x(t)$, $\forall t \geqslant 0$.

Let us consider the equation $\dot{z}_n(t) = -Cz_n(t) + D\,\mathrm{Sup}_{0 \leqslant \lambda \leqslant \tau_1} z_n(t-\lambda) + E\,\mathrm{Sup}_{0 \leqslant \lambda \leqslant \tau_2} z_n(t-\lambda) + r_n$, with the initial conditions $z_n(\theta) = z(\theta) + r_n$, $r_n > 0$. Then $z_n(t) > x(t)$, $\forall t \geqslant 0$. Moreover, the solution of this equation tends to the solution of (C.2) when the parameter r_n tends to 0. Thus, $x(t) \leqslant z(t)$ as soon as $x(\theta) \leqslant z(\theta)$. $-\mathrm{Max}(\tau_1, \tau_2) \leqslant \theta \leqslant 0$.

The asymptotic stability of the zero solution of (C.2) is easily proved using [22, 14] or [3] for example.

(2) (*case of irreducibility of $C - D - E$*): $k\mathrm{e}^{-\gamma t}$ is a solution of the Eq. (C.2), if and only if γ is an eigenvalue of $C - D\mathrm{e}^{\gamma\tau_1} - E\mathrm{e}^{\gamma\tau_2}$ and k is an eigenvector associated with it.

Let us define $A_\sigma = C - D\mathrm{e}^{\sigma\tau_1} - E\mathrm{e}^{\sigma\tau_2} = bI - P_\sigma$ ($P_\sigma \geqslant 0$), where b is the maximum value of the diagonal elements of $(C - D - E)$. Its eigenvalue with the minimum real part, $\lambda(A_\sigma)$, is equal to $b - \rho(P_\sigma)$ where $\rho(P_\sigma)$ is the spectral radius of P_σ (the maximum eigenvalue, nonnegative in the case of a nonnegative matrix).

Considering the properties of M-matrices and of matrices with nonnegative entries (see [6, 9]), it can be proved that

- As A_0 is an M-matrix, $b > \rho(P_0)$, and so $b > \rho(P_\sigma)$ for small values of $\sigma > 0$. Thus, A_σ is an M-matrix for small values of $\sigma > 0$;
- A_σ is irreducible;
- $\lambda(A_{\sigma 1}) \geqslant \lambda(A_{\sigma 2})$ if $\sigma_1 < \sigma_2$;
- $\lambda(A_0) > 0$;
- $\lambda(A_\sigma) < 0$ for large values of σ.

So $\lambda(A_\sigma) = \sigma$ has a positive solution γ.

As A_γ is an M-matrix, $A_\gamma^{-1} \geqslant 0$. As A_γ is irreducible, an eigenvector $k > 0$ of A_γ associated with the eigenvalue γ may be found (see [6, p. 383, Theorem (2,1)]).

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