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Stability domains for neutral systems

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Abstract

This paper provides local stability conditions for delay systems of neutral type, with possible uncertainties on the parameters. Positively invariant domains are constructed, that are workable estimations of the asymptotic stability domain. The stability analysis is achieved on the basis of vector comparison techniques, reducing the study to the fields of Ordinary Differential Equations and Ordinary Difference Equations.

1 Introduction

A main question in nonlinear engineering systems is to validate, in a robust way, the control laws that are generally defined on the basis of local models. Among the concrete induced problems, the evaluation of the admissible perturbations or the changes of operating points generally implies the resolution of stability with regard to the initial condition problems. For this, applying qualitative methods is not sufficient: it is needed to calculate, or estimate, the asymptotic stability domains of the operating points.

Many such studies have been (and are still) devoted to Ordinary Differential Equations. Concerning Functional Differential Equations (FDE), the direct Lyapunov's method [5] remains of basic interest: however, despite recent results [4], the construction of a Lyapunov's *function* (Razumikhin's approach [8]), or a more general *functional* (Krasovskii's approach [5]) for FDE still remains an heuristic procedure, if not an art...

Using differential inequalities and *comparison systems* is an other interesting approach, initially defined for ODE [7], and further, applied to FDE of retarded type [6], [1]. Concerning FDE of neutral type, the only existing comparison approach was based on scalar differential inequality systems [3], using so-called Degenerate Lyapunov's Functions. Very recently, the notion of Degenerate Comparison System (DCS) was briefly exposed [9].

The present work enlarges it to the problems of positive invariance and stability domains estimation: a first part presents the technical background, and a second one gives a constructive condition for a set to be a positively invariant estimation of the domain of asymptotic stability.

1.1 Notations and assumptions

The considered systems are of neutral type, described by:

$$\frac{dDx_t}{dt} = A(t, x_t)Dx_t + F_2(t, x_t) \tag{1}$$

$$Dx_t = x(t) - Bx(t - h)$$

where : h is a positive constant representing the time delay ; B is a matrix with constant coefficients ; $C = C([t_0 - h, t_0], \mathbb{R}^n)$ is the set of all continuous functions mapping $[t_0 - h, t_0]$ onto $\mathbb{R}^n$, $x(t) \in \mathbb{R}^n$ and $x_t \in C$ is the state function defined by $x_t(\theta) = x(t + \theta)$ for $\theta \in [-h, 0]$.

A and F_2 can involve uncertain (unknown) coefficients: $A(t, x_t) \in \mathbb{R}^{n \times n}$ and the function F_2 are assumed to present sufficient smoothness properties ensuring the existence of a solution of (1) (see for instance [2]). $A(t, x_t)$ is bounded for bounded x_t, and in the following $A(.)$ represents a simplified notation for $A(t, x_t)$. The solution $x_t \equiv 0$ is an equilibrium of (1), and this is guaranteed by :

$$F_2(t, 0) = 0, \quad \forall t \in \mathbb{R} \tag{2}$$

$\mathbb{R}^n$ is decomposed into the direct sum of r subspaces $\mathbb{R}^{n_i}$ ($i = 1$ to r), with $x_i \in \mathbb{R}^{n_i}$ the projection of x onto $\mathbb{R}^{n_i}$. V is some candidate Vector Lyapunov Function (VLF), where V_i is a scalar norm on $\mathbb{R}^{n_i}$

$$V : \mathbb{R}^n \to \mathbb{R}^r (r \le n), V(x) = [V_1(x_1), .., V_i(x_i), ..V_r(x_r)]^T$$

(V is said to be a Regular Vector Norm).
$\gamma_i(S)$ is the measure, or logarithmic norm [4] of a square matrix S with respect to V_i. It is assumed that F_2 verifies a boundedness-type condition on a given subset Ω of C:

$$V(F_2(t, x_t)) \le N(t)V(Dx_t), \forall x_t \in \Omega \subseteq C \tag{3}$$

where $N(t)$ is a matrix of size r with scalar continuous nonnegative coefficients.
Lastly, $|c|$ denotes the absolute value of any scalar c.

2 Background

In this section we recall some previous definitions and results given in [4] and [9].

Definition 1 *Let $g(t,.) : \mathbb{R} \times \mathbb{R}^r \to \mathbb{R}^r$ be a quasi-monotone non-decreasing function with regard to its second argument, this is, verifying the usual Ważewski conditions [6]. Then system (4) :*

$$D^+ y = g(t, y), \quad \forall t \geq t_0, \forall y \in \mathbb{R}^r \quad (4)$$

is a Degenerate Comparison System (DCS) of (1) with respect to the VLF V and the set Ω, if the following inequality is satisfied along every motion of (1)

$$D^+ V(Dx_t) \leq g(t, V(Dx_t)), \quad \forall t \geq t_0, \forall x_t \in \Omega \quad (5)$$

If $\Omega = C$, the DCS is said to be global.

Remark: In this paper, we shall only use functions g of linear time-varying type:

$$g(t, y(t)) = (M(t) + N(t))y(t) \quad (6)$$

A simple and systematic procedure for obtaining Degenerate Comparison Systems is given in [9]. Additional conditions is needed for the DCS to be a comparison system.

Lemma 1 [9]
If B has all its eigenvalues inside the unit circle, then (4) is a comparison system of (1) with regard to stability and asymptotic stability.

Theorem 1 (stability conditions) [9]
Let us note $\Phi(t_0, t)$ the transition matrix of system (4) (with (6)). If all the eigenvalues of B lie in the unit circle, and if there exists $\eta < \infty$ such that $(\forall t \geq t_0)(\|\Phi(t_0, t)\| \leq \eta)$ (respectively $\|\Phi(t_0, t)\| \to 0$ as $t \to \infty$), then the trivial zero solution of (1) is stable (respectively asymptotically stable).

3 Positive invariant sets, estimates of attraction and stability domains

Definition 2 *A subset C_i of C is said to be positively invariant with respect to (1) if any solution of (1) with initial functional conditions in C_i at time t_0 remains in C_i for all times $t \geq t_0$.*
A subset C_a is said to be a domain of attraction of the zero solution of (1) if any solution of (1) with initial conditions in C_a converges asymptotically towards zero. A subset C_s is said to be a domain of stability of the zero solution of (1) if any solution of (1) with initial conditions in C_s verifies:

$$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall \phi \in C_s)(\|\phi\| \leq \delta) \Rightarrow (\|x_t(t_0, \phi)\| \leq \varepsilon)$$

Any subset of $C_s \cap C_a$ is said to be an estimate of the asymptotic stability domain of the zero solution of (1).

Let us define $\Im(u) = \{\phi \in C : V(D\phi) \leq u\}$,
where u is a r-sized vector with positive components, such that the inclusion of $\Im(u)$ in Ω holds.

Theorem 2 *Let the system (1) verify the hypothesis (4) and let $M(t)$ and $N(t)$ be defined by (6) for a given VLF V. If there exists $\varepsilon > 0$ and a constant positive vector u such that :*

$$(M(t) + N(t))u < -\varepsilon u \quad (7)$$

then $\Im(u)$ is a positively invariant estimation of the asymptotic stability domain of the zero solution of (1).

Corollary: particular cases for condition (7)

i) *If $M(t) + N(t)$ is a Hurwitz constant matrix, then it is the opposite of an M-matrix [1], and a possible positive constant vector u is an importance eigenvector (this is an eigenvector associated with the eigenvalue of highest real part).*

ii) *If all the nonconstant coefficients of $M(t) + N(t)$ are located in an unique column or in a unique row, and if there is an $\varepsilon > 0$ such that $M(t) + N(t) + \varepsilon I_r$ is Hurwitz at any time, then a possible positive constant vector u is the importance eigenvector of $M(t) + N(t) + \varepsilon I_r$.*

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