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Stability of linear differential equations with distributed delay

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Abstract

This paper deals with the stability analysis of linear functional differential equation (FDEs) of retarded type, with both discrete and distributed delay. The main tool used for solving such problem is based on comparison principle : the behaviour of the solutions of the initial system is compared, through some regular vector Lyapunov's functions (RVLF), to the behaviour of the solution of some simpler FDE.

1 Introduction and statement of the problem

Stability and stabilization study is one of the major problems encountered in control applications. For functional differential equations, such study generally involves some Lyapunov functionals (Krasovski-i's approach [12]) or functions (Razumikhin's approach [15]), but the way to generate them is of course not clear.

An other approach based on a comparison principle has been proposed: it consists in studying a simpler system, whose stability implies the stability of the original system. This comparison approach constitutes an alternative and a development to the usual direct Lyapunov's method. It has been previously defined for ordinary differential equations (ODE)[1][13][14], and then generalized to retarded systems [2][9]. Recently, the same approach together with the concept of degenerate Lyapunov function [10][11] has been developped for neutral systems [16][17][3]. However, up to now, it was not applied to distributed delay systems.

The aim of this paper is then to complete this lack and define a comparison principle for such systems, that are often used as models for many processes in biology, economic, environment, ecology, interactions in population, ... (see for instance [4][5][7][10]).

This presentation is, firstly, developped on the basis of linear models with distributed delays, of type

$$\dot{x}(t) = Ax(t)+\sum_{i=1}^{m_1} B_i x(t-h_i)+\sum_{j=1}^{m_2}\int_{-\tau_j}^{0} K_j(s)x(t+s)ds, \quad (1)$$

$x_{t_0}(\theta) = \phi(\theta), \forall\theta \in [-h, 0]$, $h = max_{i,j}\{h_i, \tau_j\}$, h_i, τ_j constant > 0, $K_j(s)$ is a matrix with continuous entries,

where $x(t) \in R^n$ denotes the instantaneous value of the state function $x_t \in C$, $C = C([-h, 0]\ , R^n\ )$ is the set of all continuous functions mapping $[-h, 0]$ into R^n with norm $||.||_C$ defined by $||\psi||_C = \sup(||\psi(\theta)|| : \theta \in [-h, 0])$ and $||.||$ is a norm on R^n; x_t is defined by $\forall\theta \in [-h, 0], x_t(\theta) = x(t + \theta)$. In the following, $x(t; t_0, \phi)$ or $x_t(t_0, \phi)$ will denote the solution of (1) with initial condition $x_{t_0}(\theta) = \phi(\theta), \forall\theta \in [-h, 0]$.

Secondly, we apply the obtained results to derive sufficient stability conditions for some linear time invariant system with both discrete and distributed delay. For these last models, necessary and sufficient stability conditions have been defined [5][10], but in practice they need complex computations that exclude, for instance, the formal design of stabilizing controllers. Note that, the usual comparison procedure [2][16], can be generalized to nonlinear systems.

2 Background

2.1 Additional notations:

- For any $x \in R^n$ we consider a regular partition of x into $x = [x_1, .., x_i, .., x_r]^T$ (regular means that $\sum_{i=1}^{r} n_i = n$).

$V : R^n \longrightarrow R$, $V(x) = [V_1(x_1), .., V_i(x_i), .., V_r(x_r)]^T$ is some candidate of Regular Vector Lyapunov Function (RVLF), where V_i is a scalar norm on R^{n_i}.

According to the partition of x, A_{ij} denotes a $n_i \times n_j$ submatrix of any $n \times n$ matrix A partionned into

$$A = \begin{pmatrix} A_{11} & . & A_{1i} & . & A_{1r} \\ . & . & . & . & . \\ A_{i1} & . & A_{ii} & . & A_{ir} \\ . & . & . & . & . \\ A_{r1} & . & A_{ri} & . & A_{rr} \end{pmatrix}.$$

The two $r \times r$ matrices $V(A)$ and $\Gamma(A)$ are defined as follows: $V(A) = [V(A)_{ij}]$, $V(A)_{ii} = V_i(A_{ii})$ and $V(A)_{ij} = \sup \{\frac{V_i(A_{ij}x_j)}{V_j(x_j)} : V_j(x_j) \neq 0, i \neq j\}$;

$\Gamma(A) = [\Gamma(A)_{ij}]$, $\Gamma(A)_{ii} = \gamma_i(A_{ii})$ and $\Gamma(A)_{ij} = V(A)_{ij}$

- $\gamma_i(A_{ii})$ is the logarithmic norm [9] of a square matrix A_{ii} with respect to the norm V_i defined by $\gamma_i(A_{ii}) = \lim_{\Delta \to 0^+} \frac{1}{\Delta}[V_i(I_i + \Delta A_{ii}) - 1]$, I_i is the idendity matrix of order n_i.

- $\lambda(A)$, sp(A) denotes respectively an eigenvalue and the sets of all eigenvalues of A.

- $\sigma(A) = sup\{|\lambda|, \lambda \in sp(A)\}$.

- Real(λ), is the real part of the complex number λ.

- The notation $x \leq y$ for vectors or $A \leq B$ for matrices are corresponding to inequality for each corresponding entry.

- For a vector $z, |z|$ means vector constituted by absolute value of each corresponding entry of z.

Definition 1 *(***comparison principle***) A dynamic system (A) is said to be a Comparison System of a dynamic system (B) with regard to the property P (for example, stability of its zero solution), if the verification of property P for system (A) implies the same property for system (B).*

Definition 2 **(overvaluing system***) Consider the system*

$$\begin{aligned} \dot{z}(t) &= g(t, z(t), z_t), t \geq t_0, \qquad (2) \\ z(t) &\in R^r, \ z_t \in C([-h, 0]; R^r) \end{aligned}$$

where $g : R \times R^r \times C([-h, 0]; R^r) \longrightarrow R^r$.

The system (2) is said to be an overvaluing system of (1) with respect to the RVLF V

if $V(\phi(\theta)) \leq z_{t_0}(\theta), \forall \theta \in [-h, 0]$ implies $V(x(t; t_0, \phi)) \leq z(t; t_0, z_{t_0})$, $\forall t \geq t_0$.

Remark 1 *Note that in the linear cases such an overvaluing system constitutes a comparison system with regard to (asymptotic) stability. Indeed, if $V(x(t; t_0; \phi)) \leq z(t; t_0; z_{t_0}), \forall t \geq t_0$, then any property relative to stability verified by $z(t; t_0; z_{t_0})$ is verified by $x(t; t_0; \phi)$.*

3 Main results

In this section we give a way to compute an overvaluing system for the FDE with distributed delay.

Lemma 1 *Considering a RVLF V, the following inequality is verified along all solutions of (1).*

$$\begin{aligned} D^+V(x(t)) \leq\ & \Gamma(A)V(x(t)) + \sum_{i=1}^{m_1} V(B_i)V(x(t-h_i)) \\ & + \sum_{j=1}^{m_2} \int_{-\tau_j}^{0} V(K_j(s))V(x(t+s))ds. \quad (3) \end{aligned}$$

Proof: *see appendix A*

Lemma 2 *Consider a RVLF V, and suppose that the solution $z(t; t_0, \psi)$ of system (4) exists and is unique:*

$$\begin{aligned} \dot{z}(t) &= \Gamma(A)z(t) + \sum_{i=1}^{m_1} V(B_i)z(t-h_i) + \\ & \sum_{j=1}^{m_2} \int_{-\tau_j}^{0} V(K_j(s))z(t+s)ds, \qquad (4) \\ t &\geq t_0, z_{t_0}(\theta) = \psi(\theta), \forall \theta \in [-h, 0]. \end{aligned}$$

Then (4) is a comparison system of (1) with regard to (asymptotic) stability.

Proof: see appendix B

Considering the fact that

$x(t - h_i) = x(t) - \int_{t-h_i}^{t} \dot{x}(s)ds$, (1) can be rewritten as:

$$\begin{aligned} \dot{x}(t) =\ & (A + \sum_{i=1}^{m_1} B_i)x(t) - \sum_{i=1}^{m_1} B_i A \int_{-h_i}^{0} x(t+s)ds - \\ & \sum_{i=1}^{m_1} B_i \sum_{k=1}^{m_1} B_k \int_{-h_i}^{0} x(t+s-h_k)ds - \\ & \sum_{i=1}^{m_1} B_i \sum_{j=1}^{m_2} \int_{t-h_i}^{t} ds \int_{-\tau_j}^{0} K_j(s+u)x(s+u)du \\ & + \sum_{j=1}^{m_2} \int_{-\tau_j}^{0} K_j(s)x(t+s)ds. \qquad (5) \end{aligned}$$

Consequentely, one can derive another form of comparison system (equivalent of (4)) for (1), based on (5), whose delay is twice the original delay.The following theorem give different sufficient stability conditions of zero solution of (1).

Theorem 3 *The zero solution of (1) is asymptotically stable if one of the following conditions holds*

a) $\gamma(A)+\sum_{i=1}^{m_1}||B_i||+\sum_{j=1}^{m_2}\int_{-\tau_j}^{0}||K_j(s)||ds<0$;

b) $\Gamma(A)+\sum_{i=1}^{m_1}V(B_i)+\sum_{j=1}^{m_2}\int_{-\tau_j}^{0}V(K_j(s))ds$ *is the opposite of a M-matrix;*

c) $\gamma[(A+\sum_{i=1}^{m_1}B_i)]+\sum_{i=1}^{m_1}||B_iA||h_i+(\sum_{i=1}^{m_1}||B_i||h_i)(\sum_{k=1}^{m_1}||B_k||)$
$+\sum_{i=1}^{m_1}||B_i||\sum_{j=1}^{m_2}\int_{t-h_i}^{t}ds\int_{-\tau_j}^{0}||K_j(s+u)||du+\sum_{j=1}^{m_2}\int_{-\tau_j}^{0}||K_j(s)||ds<0,\ \forall\ t$;

d) there exists a r-order vector u with positive component and $\varepsilon>0$ *such that*

$[\Gamma(A+\sum_{i=1}^{m_1}B_i)]+\sum_{i=1}^{m_1}V(B_iA)h_i+$

$(\sum_{i=1}^{m_1}V(B_i)h_i)(\sum_{k=1}^{m_1}V(B_k))$
$+\sum_{i=1}^{m_1}V(B_i\sum_{j=1}^{m_2}\int_{t-h_i}^{t}ds\int_{-\tau_j}^{0}V(K_j(s+u))du+\sum_{j=1}^{m_2}\int_{-\tau_j}^{0}V(K_j(s))du]u<-\varepsilon u,\ \forall\ t$.

Consider the set defined by :

$\Omega=\{\phi\in C: V(\phi)\leq\lambda u\}$, $\lambda>0$ *scalar,* $u>0$ *a r-order vector.*

1) If condition b) holds and u is the eigenvector associated to the eigenvalue of

$\Gamma(A)+\sum_{i=1}^{m_1}V(B_i)+\sum_{j=1}^{m_2}\int_{-\tau_j}^{0}V(K_j(s))ds$ *with greatest real part then* Ω *is a positively invariant set of trajectories of (1).*

2) If u is the vector defined in condition d) then Ω *is a positively invariant set of trajectories of (1).*

Proof: see appendix C

4 Applications

4.1 Time invariant systems with discrete and distributed delays

In this part we apply previous results to derive sufficient stability conditions for some linear time-invariant delay systems. Consider the following system (6)

$$\dot{x}(t)=Ax(t)+\sum_{i=1}^{m_1}B_ix(t-h_i)+\sum_{j=1}^{m_2}C_j\int_{t-\tau_j}^{t}x(s)ds, \quad (6)$$

(6) can be rewritten as (7)

$$\begin{aligned}\dot{x}(t) &= (A+\sum_{i=1}^{m_1}B_i)x(t)-\sum_{i=1}^{m_1}B_iA\int_{-h_i}^{0}x(t+u)du\\ &-\sum_{i=1}^{m_1}B_i\sum_{k=1}^{m_1}B_k\int_{-h_i}^{0}x(t+u-h_k)du\\ &-\sum_{i=1}^{m_1}B_i\sum_{j=1}^{m_2}B_jC_j\int_{-h_i}^{0}\int_{-\tau_j}^{0}x(t+u+s)dsdu\\ &+\sum_{j=1}^{m_2}C_j\int_{-\tau_j}^{0}x(t+s)ds \quad (7)\end{aligned}$$

Then theorem 4 is a corollary of theorem 3.

Theorem 4 *The zero solution of (6) is asymptotically stable if one of the following four conditions is satisfied*

i) $\gamma(A)+\sum_{i=1}^{m_1}||B_i||+\sum_{j=1}^{m_2}||C_j||\tau_j<0$;

ii) $\Gamma(A)+\sum_{i=1}^{m_1}V(B_i)+\sum_{j=1}^{m_2}V(C_j)\tau_j$ *is the opposite of a M-matrix;*

iii) $\gamma(A+\sum_{i=1}^{m_1}B_i)+\sum_{i=1}^{m_1}||B_iA||h_i+$

$(\sum_{k=1}^{m_1}||B_k||)(\sum_{i=1}^{m_1}||B_i||h_i)+$

$(\sum_{i=1}^{m_1}||B_i||h_i)(\sum_{j=1}^{m_2}||C_j||\tau_j)+\sum_{j=1}^{m_2}||C_j||\tau_j<0$;

iv) $\Gamma(A+\sum_{i=1}^{m_1}B_i)+\sum_{i=1}^{m_1}V(B_iA)h_i+$

$(\sum_{k=1}^{m_1}V(B_k))(\sum_{i=1}^{m_1}V(B_i)h_i)+$

$(\sum_{i=1}^{m_1}V(B_i)h_i)(\sum_{j=1}^{m_2}V(C_j)\tau_j)+\sum_{j=1}^{m_2}V(C_j)\tau_j$ *is the opposite of a M-matrix.*

Example 1 *Consider the system*

$$\dot{x}(t)=Ax(t)+Bx(t-h)+C\int_{t-\tau}^{t}x(s)ds \quad (8)$$

with $A=\begin{pmatrix}-a&0\\0&-a\end{pmatrix}$; $B=\begin{pmatrix}b_1&b_2\\-b_2&b_1\end{pmatrix}$ *and* $C=\begin{pmatrix}c_1&c_2\\-c_2&c_1\end{pmatrix}$

Let set $b=\sqrt{b_1^2+b_2^2}, c=\sqrt{c_1^2+c_2^2}$; and then considering Hőlder norm $||x||=\sqrt{x_1^2+x_2^2}$, we have:

$\gamma(A+B) = -a+b_1; \gamma(A) = -a; ||BA|| = ab; ||B|| = b; ||C|| = c$.

Applying i) and iii) of theorem 4, it follows the two sufficient conditions of asymptotic stability of zero solution of (8).

1) $a > b$ and $\tau < \frac{a-b}{c}$;

2) $a - b_1 > 0, \tau < \frac{a-b_1}{c}$ and $h < \frac{a-b_1-\tau c}{b(a+b)+\tau bc}$

4.2 Time invariant systems with discrete delays

Now let consider the following class of delay system.

$$\dot{x}(t) = \sum_{i=1}^{m} B_i x(t-h_i), h_i \geq 0 \qquad (9)$$

This system can be rewritten as (10)

$$\dot{x}(t) = (\sum_{i=1}^{m} B_i)x(t) - \sum_{i=1}^{m} B_i \sum_{j=1}^{m} B_j \int_{-h_i}^{0} x(t+u-h_j)du \qquad (10)$$

Applying previous results, theorem 5 follows.

Theorem 5 *The zero solution of (9) is asymptotically stable if one of the two following conditions holds*

i) $\gamma(\sum_{i=1}^{m} B_i) + (\sum_{j=1}^{m} ||B_j||)(\sum_{i=1}^{m} ||B_i||h_i) < 0$,

ii) $\Gamma(\sum_{i=1}^{m} B_i) + (\sum_{j=1}^{m} V(B_j))(\sum_{i=1}^{m} V(B_i)h_i)$ *is the opposite of a M-matrix.*

Example 2 C*onsider (11)*

$$\dot{x}(t) = B\sum_{i=1}^{m} x(t-h_i) \qquad (11)$$

Applying theorem 5, it follows that, the zero solution of (11) is asymptotically stable, if one of the two following conditions holds

1) $0 \leq \sum_{i=1}^{m} h_i < -\frac{\gamma(B)}{||B||^2}$,

2) $\Gamma(B) + V(B)^2 \sum_{i=1}^{m} h_i$ is the opposite of a M-matrix.

4.3 Time invariant systems with distributed delays

Finally let us consider the equation of form (12)

$$\dot{x}(t) = C\int_{t-\tau}^{t} x(s)ds \qquad (12)$$

Considering the fact that $x(s) = x(t) - \int_{t-s}^{t} \dot{x}(u)du$ and $\dot{x}(u) = C\int_{u-\tau}^{u} x(v)dv$, (12) becomes

$$\dot{x}(t) = \tau Cx(t) - C^2 \int_{t-\tau}^{t} ds \int_{s}^{t} du \int_{u-\tau}^{u} x(v)dv \qquad (13)$$

Then it follows theorem 6.

Theorem 6 *The zero solution of (12) is asymptotically stable if one of the two following conditions holds*

i) $\gamma(C) + \frac{\tau^2||C^2||}{2} < 0$,

ii) $\Gamma(C) + \frac{\tau^2}{2}V(C^2)$ *is the opposite of a M-matrix.*

5 Conclusion

This paper has provided some results for testing the stability and condition of positively invariance of some bounded subsets for some linear functional differential equation of retarded type with distributed delay. The given conditions depend on the size of distributed and / or discrete delays and are simple to check. The inconvenient of these results is that they are based on majoration: that explains why the obtained conditions are sufficient, but not neccessary in general. Note that our approach is still valid in the nonlinear case if the nonlinear system can be overvalued by a linear system.

Appendix A: proof of lemma 1

Let us define for any $p \in \{1, 2, ..., r\}$ as in [9] the function $Q_p : R^{n_p} \times R^{n_p} \longrightarrow R$ defined by

$$Q_p[x_p, y_p] = \lim_{\Delta \to 0^+} \frac{1}{\Delta}[V_p(x_p + \Delta y_p) - V_p(x_p)] \qquad (14)$$

then we have [9]

$$D^+V_p(x_p(t)) = Q_p[x_p(t), \dot{x}(t)] \qquad (15)$$

$D^+V_p(x_p(t)) = Q_p[x_p(t), A_{pp}x_p(t) +$

$\sum_{k=1,k\neq p}^{r} A_{pk}xp(t) + \sum_{i=1}^{m_1}\sum_{k=1}^{r} B_{i_{pk}}x_k(t-h_i) + \sum_{j=1}^{m_2}\sum_{k=1}^{r}\int_{-\tau_j}^{0} K_j(s)_{pk}x_k(t+s)ds]$.

$= \lim_{\Delta \to 0^+} \frac{1}{\Delta}[V_p(x_p(t) + \Delta(A_{pp}x_p(t) +$

$\sum_{k=1,k\neq p}^{r} A_{pk}x_p(t) + \sum_{i=1}^{m_1}\sum_{k=1}^{r} B_{i_{pk}}x_k(t-h_i) + \sum_{j=1}^{m_2}\sum_{k=1}^{r}\int_{-\tau_j}^{0} K_j(s)_{pk}x_k(t+s)ds)) - V_p(x_p(t))]$.

Therefore we have:

$D^+V_p(x_p(t)) \leq \lim_{\Delta\to 0^+} \frac{1}{\Delta}[V_p(x_p(t)+\Delta A_{pp}x_p(t)) - V_p(x_p(t))] +$

$V_p(\sum_{k=1,k\neq p}^{r} A_{pk}x_k(t) + \sum_{i=1}^{m_1}\sum_{k=1}^{r} B_{i_{pk}}x_k(t-h_i) + \sum_{j=1}^{m_2}\sum_{k=1}^{r}\int_{-\tau_j}^{0} K_j(s)_{pk}x_k(t+s)ds)$

Remark that [9]:

$\gamma_p(A_{pp}) = \sup\{V_p(x_p(t))^{-1}Q_p[x_p(t), A_{pp}x_p(t)], x_p \in R^{n_p}, x_p \neq 0\}$

and then

$D^+V_p(x_p(t)) \leq \gamma_p(A_{pp})V_p(x_p(t)) +$

$V_p(\sum_{k=1,k\neq p}^{r} A_{pk}x_k(t) + \sum_{i=1}^{m_1}\sum_{k=1}^{r} B_{i_{pk}}x_k(t-h_i) + \sum_{j=1}^{m_2}\sum_{k=1}^{r}\int_{-\tau_j}^{0} K_j(s)_{pk}x_k(t+s)ds)$.

Considering properties of γ_p and V_p, we obtain

$D^+V_p(x_p(t)) \leq \gamma_p(A_{pp})V_p(x_p(t)) +$

$\sum_{k=1,k}^{r} V(A_{pk})V(x_k(t)) +$

$\sum_{i=1}^{m_1}\sum_{k=1}^{r} V(B_{i_{pk}})V(x_k(t-h_i)) +$

$\sum_{j=1}^{m_2}\sum_{k=1}^{r}\int_{-\tau_j}^{0} V(K_j(s)_{pk})V(x_k(t+s))ds$

This holds for all p from 1 to r, and therefore the final result is:

$$\begin{aligned} D^+V(x(t)) &\leq \Gamma(A)V(x(t)) + \sum_{i=1}^{m_1} V(B_i)V(x(t-h_i)) \\ &\quad + \sum_{j=1}^{m_2}\int_{-\tau_j}^{0} V(K_j(s))V(x(t+s))ds \quad (16) \end{aligned}$$

Appendix B: proof of lemma 2

We shall proceed by contradiction. Suppose $V(\phi(\theta)) \leq \psi(\theta)$, $\forall\, \theta \in [-h, 0]$ and

$V(x(t;t_0,\phi)) \leq z(t;t_0,\psi), \forall\ t \geq t_0$ does not hold. Then it means that there exists an instant

$t_1 = \inf\{t > t_0,\ V_i(x(t;t_0,\phi)) \geq z_i(t;t_0,\psi),\ i = 1..r :\}$ and an index $p \in \{1,2,..,r\}$ such that

$$\begin{aligned} V_p(x_p(t_1;t_0,\phi)) &= z_p(t_1;t_0,\psi) \\ V_k(x_k(t_1;t_0,\phi)) &\leq z_k(t_1;t_0,\psi), k \neq p \quad (17) \end{aligned}$$

and there exists $\varepsilon > 0$ such that

$V_p(x_p(t_1+\varepsilon;t_0,\phi)) > z_p(t_1+\varepsilon;t_0,\psi)$,

then according to (17) we have

$\frac{1}{\varepsilon}[V_p(x_p(t_1+\varepsilon;t_0,\phi)) - V_p(x_p(t_1;t_0,\phi))] > \frac{1}{\varepsilon}[z_p(t_1+\varepsilon;t_0,\psi) - z_p(t_1;t_0,\psi)]$,

and taking the limit as $\varepsilon \longrightarrow 0^+$ we obtain the inequality

$$D^+V_p(x_p(t_1;t_0,\phi)) > (\frac{z_p(t;t_0,\psi)}{dt})_{t=t_1} \quad (18)$$

But, from lemma 1, at the instant t_1 we have the inequality

$D^+V_p(x_p(t_1)) \leq \gamma_p[A_{pp}]V_p(x_p(t_1)) +$

$\sum_{k=1,k}^{r} V(A_{pk})V(x_k(t_1)) + \sum_{i=1}^{m_1}\sum_{k=1}^{r} V(B_{i_{pk}})V(x_k(t_1 - h_i)) + \sum_{j=1}^{m_2}\sum_{k=1}^{r}\int_{-\tau_j}^{0} V(d_sK_j(s)_{pk})V(x_k(t_1+s))$

As all the off-diagonal entries of matrix $\Gamma(A)$ and all entires of $V(B_i)$ and $V(K_j(s))$ are non negative by construction, relations (17) implies:

$D^+V_p(x_p(t_1)) \leq \gamma_p[A_{pp}]V_p(z_p(t_1)) +$

$\sum_{k=1,k}^{r} V(A_{pk})V(z_k(t_1)) + \sum_{i=1}^{m_1}\sum_{k=1}^{r} V(B_{i_{pk}})V(z_k(t_1 - h_i)) + \sum_{j=1}^{m_2}\sum_{k=1}^{r}\int_{-\tau_j}^{0} V(d_sK_j(s)_{pk})V(z_k(t_1+s))$

This means that

$$D^+V_p(x_p(t_1;t_0,\phi)) \leq (\frac{z_p(t;t_0,\psi)}{dt})_{t=t_1} \quad (19)$$

which contradicts inequality (18).

Appendix C: proof of theorem 3

We only consider here the proof of the point d) ; the other ones are similar. It is sufficient to prove that zero solution of the comparison system (equivalent of (4)), derived on basis of equation (5), is asymptotically stable. Consider this system and suppose that assumption d) holds. Then, consider the following function $v(z(t))$

$$v(z(t)) = max_{1\leq i\leq r}\{\frac{|z_i(t)|}{u_i}\} \quad (20)$$

For each t, there exists an index k such that

$v(z(t)) = \frac{|z_k(t)|}{u_k}$ and

$\frac{d^+v(z(t))}{dt} \leq \frac{1}{u_k}[\Gamma[(A + \sum_{i=1}^{m_1} B_i)]|z(t)| +$

$\sum_{i=1}^{m_1} V(B_iA)\int_{-h_i}^{0}|z(t+s)|ds +$

$\sum_{i=1}^{m_1} V(B_i)\sum_{k=1}^{m_1} V(B_k)\int_{-h_i}^{0}|z(t+s-h_k)|ds + \sum_{i=1}^{m_1} V(B_i)\sum_{j=1}^{m_2}\int_{t-h_i}^{t} ds\int_{-\tau_j}^{0} V(K_j(s+u))|z(s+u)|du + \sum_{j=1}^{m_2}\int_{-\tau_j}^{0} V(K_j(s))|z(t+s)|ds]_k$

Along the trajectories verifying $|z_j(t+s)| \leq |z_j(t)|$ for $s \in [-h, 0]$, we have

$\frac{d^+v(z(t))}{dt} \leq \frac{1}{u_k}[\Gamma[(A+\sum_{i=1}^{m_1} B_i)]|z(t)| +$

$\sum_{i=1}^{m_1} V(B_iA)\int_{-h_i}^{0}|z(t)|ds +$

$\sum_{i=1}^{m_1} V(B_i)\sum_{k=1}^{m_1} V(B_k)\int_{-h_i}^{0}|z(t)|ds +$

$\sum_{i=1}^{m_1} V(B_i)\sum_{j=1}^{m_2}\int_{t-h_i}^{t} ds\int_{-\tau_j}^{0} V(K_j(s+u))|z(t)|du + \sum_{j=1}^{m_2}\int_{-\tau_j}^{0} V(K_j(s))|z(t)|ds]_k$

Then, according to definition of $v(z(t))$,

$\frac{d^+v(z(t))}{dt} \leq \frac{1}{u_k}[(\Gamma[(A+\sum_{i=1}^{m_1} B_i)] + \sum_{i=1}^{m_1} V(B_iA)h_i + (\sum_{i=1}^{m_1} V(B_i)h_i)\sum_{k=1}^{m_1} V(B_k) +$

$\sum_{i=1}^{m_1} V(B_i)\sum_{j=1}^{m_2}\int_{t-h_i}^{t} ds\int_{-\tau_j}^{0} V(K_j(s+u))ds + \sum_{j=1}^{m_2}\int_{-\tau_j}^{0} V(K_j(s))ds)u]_k v(z(t))$

or

$$\frac{d^+v(z(t)}{dt} < -\varepsilon v(z(t)) \qquad (21)$$

This implies $\frac{d^+v(z(t)}{dt} < -\varepsilon v(z(t))$, so $v(z(t))$ is a Lyapunov-Razumikhin [15] function, and the conclusion follows.

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