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Estimation of the Asymptotic Stability Domain and Asymptotic Behaviour for Nonlinear Neutral Systems

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Abstract: *This paper deals with the estimation of both stability domain and asymptotic behaviours of a class of nonlinear neutral functional differential equations (FDE). The approach for solving such a complex problem is based on comparison methods: the solutions of the original system are compared (by upper-bounding) to the behaviours of some more simple ordinary differential equation (ODE). It is combined with the use of regular vector Lyapunov's functions, that appear to yield less conservative results than usual scalar norms or Lyapunov's functions.*

Keywords: asymptotic analysis, attractors, delay, differential equations, neutral systems, nonlinear systems, stability, stability domains, time-delay.

Résumé: *Ce travail concerne l'estimation à la fois des domaines de stabilité et des comportements asymptotiques d'une classe d'équations différentielles fonctionnelles non-linéaires de type neutre. Il est montré comment un problème aussi complexe peut être étudié à l'aide de méthodes de comparaison: les solutions du système original sont comparées (par majoration) aux comportements d'un système d'équations différentielles ordinaires plus simple à analyser. L'outil principal utilisé est les fonctions vectorielles de Lyapunov, qui permettent d'obtenir des résultats moins conservatifs que l'approche scalaire.*

mots-cléfs: analyse asymptotique, attracteurs, domaine de stabilité, systèmes neutres, systèmes non linéaires, stabilité, retards.

1. INTRODUCTION

Comparison principle for FDE is a general way of analysing stability of complex time-delay systems [8], that can be combined together with the concept of vector Lyapunov's functions initially defined for ODE (Bellman, 1962; Matrosov, 1962; Perruquetti *et al*, 1995). However, the general approach is often difficult to apply. By using a more special class of regular vector functions, many calculable results has been obtained (Dambrine, 1994; Dambrine and Richard, 1994; Goubet *et al*, 1996): but, they were limited to *retarded* systems (possibly nonlinear and with variable delay). On an other side, the case of *neutral* systems was studied in (Kolmanovskii and Nosov, 1979, 1986) by introducing the notion of (scalar) *degenerate* Lyapunov's functions. Very recently (Tchangani *et al*, 1996b), it was enlarged to such vector functions. All these results allowed to obtain sufficient stability conditions.

Besides, once an equilibrium is proved to be stable, engineering practice also needs to know something about its stability domain (for instance, in order to characterize the size of perturbations that can be rejected). Moreover, when the equilibrium is possibly unstable, it is also interesting to have informations about the boundedness (and size) of the asymptotic behaviours. Such questions are generally complex: in (Tchangani *et al*, 1996a), the authors estimated the stability domain of the only zero solution for some neutral systems. The present work considers, in addition, the boundedness of the asymtotic behaviours.

Let R^n be the n-dimensional real vector space with a norm $|\,.\,|$, and ρ the associated distance: $\rho(x, y) = |x-y|$. $C = C([-h, 0], R^n)$ is the set of all continuous functions mapping $[-h, 0]$ onto R^n where h is a positive constant (time delay). The norm $\|.\|$ of a function $\psi \in C$ is $\|\psi\| = \sup\{|\psi(\theta)| : \theta \in [-h, 0]\}$.

The paper considers FDE of the following neutral type:

$$\frac{d}{dt} D(t, x_t, d(t)) = F(t, x_t, d(t)), \quad t \geq t_0, x_{t0} = \varphi \in C \quad (1.1)$$

$$D(t, x_t, d(t)) = x(t) - B(t, x_t, d(t)).$$

It is assumed that the operator $D \in C(R \times C \times R^m, R^n)$ has a continuous time derivative satisfying (1.1) for each $t \in R$. $F, B \in C(R \times C \times R^m, R^n)$; $x(t) \in R^n$ denotes the instantaneous value of the state function $x_t \in C$ defined by $x_t(\theta) = x(t+\theta)$ $\forall\, \theta \in [-h, 0]$.

$d(t) \in D \subseteq R^m$ represents some disturbances.

It is assumed that (1.1) satisfies the conditions of existence and uniqueness of the solution (see (Hale,1977) for this theory).

$x(t; t_0, \varphi)$ or $x_t(t_0, \varphi)$ will denote the solution of (1.1) with initial state function $x_{t0} = \varphi$.

Additional notations

- For any $x \in R^n$ we consider a partition of x into $x^T = [x_1^T, ..., x_i^T, ... x_r^T]$ where $x_i \in R^{n_i}$ such that $\sum_{i=1}^{r} n_i = n$, this partition is said to be regular.

- $V : R^n \to R^r$ $(r \leq n)$

$V(x) = [V_1(x_1), .., V_i(x_i), .. V_r(x_r)]^T$ is some candidate of Regular Vector Lyapunov Function (RVLF), where V_i is a scalar norm on R^{n_i}.

According to the partition of x, a n×n matrix A is partionned into

$$A = \begin{pmatrix} A_{11} & . & A_{1i} & . & A_{1r} \\ . & . & . & . & . \\ A_{i1} & . & A_{ii} & . & A_{ir} \\ . & . & . & . & . \\ A_{r1} & . & A_{ri} & . & A_{rr} \end{pmatrix}$$

and the two r×r matrices V(A) and Γ(A) are defined as follows :

$V(A) = [V(A)_{ij}]$ with

$$V(A)_{ij} = \sup_{x_j : V_j(x_j) \neq 0} \left\{ \frac{V_i(A_{ij} x_j)}{V_j(x_j)} \right\}$$ (that is the norm of the matrix A with respect to the norms V_i and V_j),

$\Gamma(A) = [\Gamma(A)_{ij}]$ with $\Gamma(A)_{ii} = \gamma_i(A_{ii})$ and

$$\Gamma(A)_{ij} = \sup_{x_j : V_j(x_j) \neq 0} \left\{ \frac{V_i(A_{ij} x_j)}{V_j(x_j)} \right\}, \text{ for } i \neq j.$$

- $\gamma_i(A)$ is the logarithmic norm of the square matrix A with respect to the norm V_i.
- $\lambda(A)$, sp(A) denotes respectively an eigenvalue and the sets of all eigenvalues of A.
- $\sigma(A) = \sup\{|\lambda|, \lambda \in sp(A)\}$.
- Real(λ), is the real part of the complex number λ.
- the notation $x \leq y$ where $x, y \in R^S$ (respect. $x > y$) means $x_i \leq y_i$ for i=1 to s (respect. $x_i > y_i$) and $A \leq B$ (respect. $A > B$) where A and B are two s×s matrices means $A_{ij} \leq B_{ij}$ for i=1 to s and j=1 to s (respect. $A_{ij} > B_{ij}$).

2. STABILITY: ASSUMPTIONS, DEFINITIONS

Some hypotheses on the structure of the functional F in (1.1) are needed before establishing the main results. In this part we consider that :

- either F explicitely depends on D :

$$F(t, \psi, d(t)) = F_0(t, D(t, \psi, d(t)), d(t)), \quad t \geq t_0, \psi \in C, \quad (2.1)$$

- or it can be decomposed as :

$$F(t, \psi, d(t)) = F_0(t, D(t, \psi, d(t)), d(t)) + F_1(t, \psi, d(t)), \quad t \geq t_0, \psi \in C, \quad (2.2)$$

and F_0 is supposed to have continuous partial derivative with respect to its second argument that is denoted f_0.

It is also assumed that F_0, verifies $F_0(t, 0, d(t)) \equiv 0$.

The last assumption is that F_1 verifies

$$V(F_1(t, \psi, d(t))) \leq N(t) V(D(t, \psi, d(t))), \quad t \geq t_0, d(t) \in D, \psi \in \Omega \subseteq C \quad (2.3)$$

where N(t) is a matrix with nonnegative coefficients and Ω is a bounded subset of C.

Definition 1 (Kolmanovskii and Nosov, 1979, 1986)
The solution $x(t) \equiv 0$ of (1.1) is

a) stable if for any t_0 and any $\varepsilon > 0$ we can find $\delta(t_0, \varepsilon) > 0$ such that $|x(t;t_0,\varphi)| \leq \varepsilon$ when $t \geq t_0$ if only $\|\varphi\| \leq \delta(t_0, \varepsilon)$

b) asymptotically stable if it is stable and in addition, there is a set $C_a \subseteq C$, neighbourhood of 0 such that $\lim_{t\to\infty} x(t;t_0,\varphi) = 0$ for all $\varphi \in C_a$.

The set C_a is called the attraction domain of the trivial solution.

The study of stability of the zero solution of (1.1) is connected to some characteristics of the operator D (Hale, 1977; Kolmanovskii and Nosov, 1979, 1986). Here we consider the notion of f-stability of this operator defined in (Kolmanovskii and Nosov, 1979, 1986) .

Consider the difference inequality

$$|D(t, y_t, d(t))| \leq f(t), \; y_{t0} = \varphi, \quad (2.4)$$

where f is a nonnegative scalar function and $\varphi \in C$.

$y(t;\, t_0;\, \varphi)$ denotes the solution of the difference inequality (2.4) with initial condition $y_{t0} = \varphi$.

Definition 2 (Kolmanovskii and Nosov, 1979, 1986)
The solution $y(t) \equiv 0$ of the difference inequality (2.4) is

a) f-stable if for any t_0 and any $\varepsilon > 0$ there exists a $\delta(t_0, \varepsilon) > 0$ such that $|y(t;t_0,\varphi)| \leq \varepsilon$ for all $t \geq t_0$ under all initial conditions and functions f such that

$$\|\varphi\| \leq \delta(t_0, \varepsilon), \text{ and } \sup_{t\geq 0} f(t) \leq \varepsilon$$

b) asymptotically f-stable if it is f-stable and, in addition, $\lim_{t\to\infty} y(t;t_0,\varphi) = 0$ for all φ in a subset of C and for every function f such that $f(t) \to 0$ as $t \to \infty$

c) f-bounded if a bounded solution $y(t;t_0,\varphi)$

corresponds to each bounded function f.
These notions are said to be uniform if δ does not depend on t_0.

The main tool used in this paper is the comparison principle (Dambrine, 1994; Lakshmikantham and Leela, 1979) and the notion of degenerate comparison system (Tchangani *et al*, 1996b).

Definition 3 (Tchangani *et al*, 1996b): A dynamic system (A) is said to be a Comparison System of a dynamic system (B) with regard to (asymptotic) stability, if the (asymptotic) stability of the zero solution of (A) implies the (asymptotic) stability of the zero solution of (B).
The well known Lyapunov's functions represent some general distance functions between the trajectories and an equilibrium. When these functions are defined relatively to some operator of the state function (the conditions of usual definition of the distance are not verified), they are called degenerate Lyapunov functions (Kolmanovskii and Nosov, 1979, 1986). The corresponding notion of degenerate comparison system with respect to some degenerate regular vector Lyapunov's function (DRVLF) was defined in (Tchangani *et al*, 1996a, b):

Definition 4 (Tchangani *et al*, 1996a, b) :
Let $g(t,.)\ RxR^r \rightarrow R^r$ be a quasimonotone nondecreasing function with regard to its second argument, this is, verifying the usual Ważewski conditions (Perruquetti *et al*, 1995); then system (2.5) :

$$D^+y(t) = g(t,y(t))\ \forall t \geq t_0,\ \forall\ y \in R^r \tag{2.5}$$

is a Degenerate Comparison System (DCS) of (1.1) with respect to the RVLF V and the set Ω, if the following inequality is satisfied along every motion of (1.1) :

$$D^+V(D(t, x_t, d(t))) \leq g(t,V(D(t, x_t, d(t))) \quad \forall\ t \geq t_0, d(t) \in D, x_t \in \Omega \tag{2.6}$$

where D^+ represents the right upper Dini derivative (Lakshmikantham and Leela, 1979). If Ω = C, the degenerate comparison system is said to be global.

It is shown (Kolmanovskii and Nosov, 1979, 1986) that when (1.1) satisfies (2.1) and the operator D is f-stable (asymptotically f-stable) then if the zero solution of ordinary differential equation (ODE) (2.7) is stable (asymptotically stable) then the zero solution of (1.1) is stable (asymptotically stable)

$$\dot{y} = F_0(t,y(t), d(t)) \tag{2.7}$$

The following results give some conditions of stability and a way of obtaining some estimation of the stability domains for some FDE of neutral type satisfying the conditions (2.1) or (2.2).

Theorem 1: Suppose the following conditions are satisfied :
1) the operator D is f-stable (asymp. f-stable) ;
2) there is a subset Ω of C and a matrix $M_0(t)$ such that

$$\Gamma(f_0(t, D(t, \psi, d(t)), d(t))) \leq M_0(t), \quad t \geq t_0, d(t) \in D, x_t \in \Omega \,; \tag{2.8}$$

then the ODE

$$\dot{z} = M(t)z(t) \tag{2.9}$$

(where M(t) represents $M_0(t)$ in case of a decomposition (2.1) or $M_0(t)$+N(t) if (2.2) holds)
is a local comparison system of (1.1)-(2.1) or (1.1)-(2.2)-(2.3) with regard to (asymptotic) stability with respect to RVLF V and the subset Ω ; the comparison system is said to be global if Ω = C.
Moreover, if there exists $\varepsilon > 0$ and a positive constant vector u such that

$$M(t)u \leq -\varepsilon u \tag{2.10}$$

then the set $\Im(u)$ defined by

$$\Im(u) = \{\varphi \in C, V(D(t_0, \varphi, d(t_0))) \leq \lambda u\} \subseteq \Omega \tag{2.11}$$

where λ is a positive number the greatest possible, is a positively invariant estimation of the asymptotic stability domain of the zero solution of (1.1)-(2.1) or (1.1)-(2.2)-(2.3).

Proof :
point i) It is shown (Tchangani *et al*, 1996b) that (2.9) is a DCS of (1.1) so

$$V(D(t, x_t(t_0, \varphi), d(t))) \leq z(t; t_0, V(D(t_0, \varphi, d(t_0))))$$

where $x_t(t_0, \varphi)$ is the solution of (1.1) with initial condition φ and $z(t; t_0, V(D(t_0, \varphi)))$ the solution of (2.9) with initial condition $V(D(t_0, \varphi))$;
remark that since the different norms of a finite-dimensional space are equivalent, there are constants c_j such that

$$|D(t, x_t(t_0, \varphi), d(t))| \leq \sum_{j=1}^{r} c_j V_j(D(t, x_t(t_0, \varphi), d(t))) \leq \sum_{j=1}^{r} c_j z_j(t; t_0, V(D(t_0, \varphi, d(t_0))))$$

and as the operator D is f-stable (asymptotically f-stable), conclusion i) yields.
point ii) : Suppose $\varphi \in \Im(u)$; let us proceed by contradiction, and suppose that $x_t(t_0,\varphi)$ (the solution of (1.1) with $x_{t_0} = \varphi$) does not remain in $\Im(u)$. Then let $t_1 = \inf\{t \geq t_0 : V(D(t, x_t(t_0, \varphi), d(t))) > \lambda u\}$. According to the definition of t_1, there is an integer $k \in \{1,2,..,r\}$ such that

$$V_k(D(t_1, x_{t1}(t_0,\varphi), d(t_1))) = \lambda u_k \text{ and } V_j(D(t_1, x_{t1}(t_0,\varphi), d(t_1))) \leq \lambda u_j\,, j \neq k \tag{2.12}$$

since, at least one component (k) of $x_{t1}(t_0,\varphi)$ reaches the boundary of $\Im(u)$ at t_1), and there exists $\Delta t > 0$ such that

$$V_k(D(t, x_t(t_0,\varphi), d(t))) > \lambda u_k\,, \text{for } t \in (t_1,t_1+\Delta t) \tag{2.13}$$

But, from (2.12) and (2.9)

$$D^+V_k(D(t_1, x_{t1}(t_0,\varphi), d(t_1))) \leq M(t_1)_{kk}V_k(D(t_1, x_{t1}(t_0,\varphi), d(t_1))) + \sum_{j\neq k} M(t_1)_{kj}V_j(D(t_1, x_{t1}t_0,\varphi), d(t_1)) \tag{2.14}$$

As $M(t)_{kj} \geq 0$, according to (2.12), we obtain

$$D^+V_k(D(t_1, x_{t1}(t_0,\varphi), d(t_1))) \leq M(t_1)_{kk}\lambda u_k + \sum_{j=1}^{r} M(t_1)_{kj}\lambda u_j = \lambda[M(t_1)u]_k < 0 \tag{2.15}$$

so $\frac{d}{dt}$ $[V_k(D(t_1, x_{t1}(t_0,\varphi), d(t_1)))] \leq -\varepsilon u_k < 0$, which means that the function $t \rightarrow V_k(D(t, x_t(t_0,\varphi), d(t)))$ is strictly decreasing on a neighbourhood of t_1 and so the inequality (2.13) cannot hold when (2.12) is satisfied. This proves that $x_t(t_0,\varphi)$ remains in $\Im(u)$.

corollary (Tchangani *et al*, 1996a) :

i) If M(t) is a Hurwitz constant matrix, then it is the opposite of an M-matrix (Dambrine, 1994), and a possible positive constant vector u is an importance eigenvector (this is the eigenvector associated with the eigenvalue of highest real part).

ii) If all the nonconstant coefficients of M(t) are located in a unique colunm or in a unique row and there exists $\varepsilon > 0$ such that $M(t) + \varepsilon I_r$ is Hurwitz at any time, then a possible positive constant vector u is the importance eigenvector of $M(t) + \varepsilon I_r$ (Dambrine, 1994).

A procedure in three points can be given for investigating the problem of estimation of a positive invariant asymptotic stability domain for the zero solution of (1.1) with condition (2.1) or (2.2)-(2.3).

1) choose a RVLF V and find a subset Ω of C such that (1.1) admits a DCS relative to Ω and V (computation of M(t)) ;

2) test the stability condition of DCS and if it is positive, calculate a vector u such that $M(t)u < 0$;

3) $\Im(u) = \{\varphi \in C,\ V(D(t_0, \varphi, d(t_0))) \leq \lambda u\} \subseteq \Omega$.

3. ASYMPTOTIC BEHAVIOUR STUDY

In this section, we assume that F satisfies the decomposition (2.2) recalled below :

$$F(t, \psi, d(t)) = F_0(t, D(t, \psi, d(t)), d(t)) + F_1(t, \psi, d(t)) \quad (3.1)$$

with the same conditions on F_0 as in section 2 and F_1 verifies the following condition :

$$V(F_1(t, \psi, d(t))) \leq q, \quad \forall\ t \geq t_0,\ d(t) \in D,\ \psi \in \Omega \subseteq C \quad (3.2)$$

where q is a vector with positive components and the operator D verifies:

$$\sup_{t\geq t_0, d(t)\in D} V(B(t, \psi, d(t))) \leq B_0V(\psi(0)) + B_1V(\psi(-h)),\ \forall\ \psi \in \Omega \quad (3.3)$$

where B_0 and B_1 are $r\times r$ constant matrices with nonnegative coefficients.

In this case, it is not possible to derive a stability condition for the zero solution by applying the previous results, but it is possible to estimate behaviours such as limit-cycles. Let A be a subset of R^n and define C(A) by

$$C(A) = \{\varphi \in C : \varphi(\theta) \in A \text{ for } \theta \in [-h, 0]\}$$

Definition 5 : A subset C(A) of C is said to be an attractor of (1.1) if any solution $x(t; t_0,\varphi)$ of (1.1) converges asymptotically towards A that is $\lim_{t\to\infty} \rho(x(t;t_0,\varphi), A) = 0$ for any $\varphi \in C_a$ where $\rho(x(t;t_0,\varphi), A) = \inf_{y\in A} |x(t;t_0,\varphi)-y|$; the set C_a is called the domain of attraction of C(A).

Theorem 2 : Assume the following conditions :

1) there exists M_1, opposite of a M-matrix, such that

$$\Gamma(f_0(t, D(t, x_t, d(t)), d(t)) \leq M_1,\quad t \geq t_0,\ d(t) \in D,\ x_t \in \Omega\ ; \quad (3.4)$$

2) the operator D satisfies (3.3) ;

3) $B_0 + B_1$ has all its eigenvalues inside the unit circle.

Consider

$$A_0 = \{y \in R^n : V(y) \leq -(I_r - B_0 - B_1)^{-1}M_1^{-1}q\} \quad (3.5)$$

then $C(A_0)$ is a stable attractor of (1.1)-(2.2)-(3.2)-(3.3) and for any vector u with positive constant coefficients satisfying the following conditions

a) $u \geq q$ (3.6)

b) $M_1u < 0$ (3.7)

c) $\Im(u) = \{\varphi \in C,\ (D(t_0, \varphi, d(t_0))) \leq u\} \subseteq \Omega$

the set $\Im(u)$ is an estimation of its positively invariant domain of attraction.

Proof : It is known (Tchangani *et al*, 1996b) that the nonhomogeneous systems (3.8)

$$\dot{z}(t) = M_1 z(t) + q \quad (3.8)$$

is such that

$V(D(t, x_t(t_0, \varphi), d(t))) \leq z(t; t_0, V(D(t_0, \varphi, d(t_0))))$, $t \geq t_0$; considering the definition of operator D we obtain

$$V(x(t; t_0, \varphi)) \leq V(B(t, x_t, d(t))) + z(t; t_0, V(D(t_0, \varphi, d(t_0)))),\ t \geq t_0$$

and applying (3.3) we obtain:

$$V(x(t; t_0, \varphi)) \leq B_0V(x(t; t_0, \varphi)) + B_1V(x(t-h; t_0, \varphi)) + z(t; t_0, V(D(t_0, \varphi, d(t_0)))),\ t \geq t_0 \quad (3.9)$$

notice that $z(t; t_0, V(D(t_0, \varphi, d(t_0))))$ is given by

$$z(t; t_0, V(D(t_0, \varphi, d(t_0)))) = \exp[M_1(t-t_0)]\{V(D(t_0, \varphi, d(t_0))) + M_1^{-1}q\} - M_1^{-1}q$$

and as M_1 is the opposite of a M-matrix, $z(t; t_0, V(D(t_0, \varphi, d(t_0))))$ converges asymptotically towards $-M_1^{-1}q$; so from (3.9) we deduce

$$\lim_{t\to\infty} V(x(t; t_0, \varphi)) \leq B_0 \lim_{t\to\infty} V(x(t; t_0, \varphi)) + B_1 \lim_{t\to\infty} V(x(t-h; t_0, \varphi)) - M_1^{-1}q \quad (3.10)$$

and so: $(I_r - B_0 - B_1) \lim_{t\to\infty} V(x(t; t_0, \varphi)) \leq -M_1^{-1}q$

According to condition 3), the matrix $(I_r - B_0 - B_1)$ is a M-matrix and so $(I_r - B_0 - B_1)^{-1}$ exists and has all its coefficients nonnegative ; it follows: $\lim_{t\to\infty} V(x(t; t_0, \varphi)) \leq -(I_r - B_0 - B_1)^{-1}M_1^{-1}q$,

which means that $x_t(t_0, \varphi)$ converges asymptotically towards $C(A_0)$; its stability comes obsviously from the stability of the the equilibrium $-M_1^{-1}q$ of (3.8); this implies the firts part of conclusion. Now, it leave to show that $\Im(u)$ is positively invariant for (1.1) ; this can be done in adapting the proof of point ii) of theorem 1.

In order to illustrate these results we consider two examples.

4. EXAMPLES

Example 1: Let consider the system

$$\frac{dDx_t}{dt} = \begin{pmatrix} -3+f_1(t, x_t, d(t)) & f_2(t, x_t, d(t)) \\ f_3(t, x_t, d(t)) & -4+f_4(t, x_t, d(t)) \end{pmatrix} Dx_t + \begin{pmatrix} g_1(t, x_t, d(t)) \\ g_2(t, x_t, d(t)) \end{pmatrix}$$

$$Dx_t = x(t) - \begin{pmatrix} \gamma_1 & \gamma_2 \\ 0 & 0 \end{pmatrix} x(t-h),\ t \geq t_0,\ x_{t_0} = \varphi \qquad (4.1)$$

with $x(t) = [x_1(t), x_2(t)]^T \in R^2$

Hypotheses

For all $t \geq t_0$, $d(t) \in D$, $x_t \in C$

$|f_i(t, x_t, d(t))| \leq 1,$ (4.2)

$|g_i(t, x_t, d(t))| \leq \alpha_i(t)\ |x_2(t)|,$ (4.3)

$|\gamma_1| < 1$ (4.4)

This last condition guarantees the f-stability of operator D.

This system is not exactly in form (2.2) but let us denote

$$F_0(t, x_t, d(t)) = \begin{pmatrix} -3+f_1(t, x_t, d(t)) & f_2(t, x_t, d(t)) \\ f_3(t, x_t, d(t)) & -4+f_4(t, x_t, d(t)) \end{pmatrix} Dx_t$$

and

$$F_1(t, x_t, d(t)) = \begin{pmatrix} g_1(t, x_t, d(t)) \\ g_2(t, x_t, d(t)) \end{pmatrix}$$

By using the RVLF $V(x) = [|x_1|, |x_2|]^T$, it yields

$$V(F_1(t, x_t)) = \begin{pmatrix} |g_1(t, x_t, d(t))| \\ |g_2(t, x_t, d(t))| \end{pmatrix} \leq \begin{pmatrix} \alpha_1(t)|x_2(t)| \\ \alpha_2(t)|x_2(t)| \end{pmatrix} = \begin{pmatrix} 0 & \alpha_1(t) \\ 0 & \alpha_2(t) \end{pmatrix} V(Dx_t) \quad (4.5)$$

so (2.3) is satisfied with $N(t) = \begin{pmatrix} 0 & \alpha_1(t) \\ 0 & \alpha_2(t) \end{pmatrix}$

The matrix $M_0(t)$ is obtained by establishing the differential inequality (Tchangani *et al*, 1996b):

$$D^+V(Dx_t) \leq V(F_1(t, x_t)) + \begin{pmatrix} -3+f_1(t, x_t, d(t)) & |f_2(t, x_t, d(t))| \\ |f_3(t, x_t, d(t))| & -4+f_4(t, x_t, d(t)) \end{pmatrix} V(Dx_t) \quad (4.6)$$

Considering (4.2) we obtain

$$D^+V(Dx_t) \leq \begin{pmatrix} -2 & 1 \\ 1 & -3 \end{pmatrix} V(Dx_t) + N(t)V(Dx_t)$$

so $M_1(t) = \begin{pmatrix} -2 & 1 \\ 1 & -3 \end{pmatrix}$ and at last, the parameters of the DCS of the system (4.1) are for $\Omega = C$:

$$M(t) = M_0(t) + N(t) = \begin{pmatrix} -2 & 1+\alpha_1(t) \\ 1 & -3+\alpha_2(t) \end{pmatrix} \qquad (4.7)$$

As all the nonconstant coefficients of M(t) are located in a unique column, the stability condition of (4.1) is that M(t) is Hurwitz, i.e :

$$\begin{cases} \alpha_1(t) > 0 \\ 0 < \alpha_2(t) < 5 \\ \alpha_1(t) + 2\alpha_2(t) < 5 \end{cases} \quad \text{for any time.} \qquad (4.8)$$

Example of a positively invariant asymptotic stability domain estimation

For sake of simplicity, we consider the case $\alpha_1(t) \equiv \alpha_2(t) \equiv 1$

then $M = \begin{pmatrix} -2 & 2 \\ 1 & -2 \end{pmatrix}$

The importance eigenvalue of M is $-2+\sqrt{2}$ and an eigenvector associated with it is $[\sqrt{2}, 1]^T$, so the set

$\Im(u) = \{\varphi \in C : V(D\varphi) \leq \lambda[\sqrt{2}, 1]^T\}$ (4.9)

is a positively invariant estimation of the asymptotic stability domain of zero solution of (4.1).

Example 2

We consider one more time system (4.1), but now we suppose that the operator D is defined by :

$D(t, x_t, d(t)) = x(t) - B(t, x_t, d(t)),$ (4.10)

with $B(t, x_t, d(t)) = [b_1(t, x_t, d(t));\ b_2(t, x_t, d(t))]^T$

where the functionals b_1 and b_2 verify :

$|b_1(t, \psi, d(t))| \leq \beta_{11}|\psi_1(0)| + \beta_{12}|\psi_2(0)| + \gamma_{11}|\psi_1(-h)| + \gamma_{12}|\psi_2(-h)|$

$|b_2(t, \psi, d(t))| \leq \beta_{21}|\psi_1(0)| + \beta_{22}|\psi_2(0)| + \gamma_{21}|\psi_1(-h)| + \gamma_{22}|\psi_2(-h)|))$

for $t \geq t_0$, $d(t) \in D$, $\psi \in \Omega \subseteq C$. (4.11)

Here we assume that following conditions are satisfied for $t \geq t_0$, $d(t) \in D$, $x_t \in \Omega \subseteq C$

$|f_i(t, x_t, d(t))| \leq 1,$ (4.12)

$|g_i(t, x_t, d(t))| \leq \alpha_i(t)$ (with $0 < \alpha_i(t) \leq 1$). (4.13)

In this case, the parameters of the nonhomogeneous DCS are :

$$M_1 = \begin{pmatrix} -2 & 1 \\ 1 & -3 \end{pmatrix} q = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, -M_1^{-1}q = \begin{pmatrix} 4/5 \\ 3/5 \end{pmatrix},$$

it is obsvious that M_1 is the opposite of an M-matrix and its importance eigenvalue is $-\frac{5}{2} + \frac{\sqrt{5}}{2}$ and one eigenvector associated with it is $[\frac{1+\sqrt{5}}{2}\ ;\ 1]^T$.

$$B_0 + B_1 = \begin{pmatrix} \beta_{11}+\gamma_{11} & \beta_{12}+\gamma_{12} \\ \beta_{21}+\gamma_{21} & \beta_{22}+\gamma_{22} \end{pmatrix}$$

For simplicity, we suppose $\beta_{12} = \gamma_{12} = \beta_{21} = \gamma_{21} = 0$

then conditions 3) of theorem 2 are equivalent to:

$$\begin{cases} 0 \le \beta_{11} + \gamma_{11} < 1 \\ 0 \le \beta_{22} + \gamma_{22} < 1 \end{cases}$$

and

$$-(I_2 - B_0 - B_1)^{-1} M_1^{-1} q = \begin{pmatrix} \frac{4}{5(1-\beta_{11}-\gamma_{11})} \\ \frac{3}{5(1-\beta_{22}-\gamma_{22})} \end{pmatrix}$$

So, the set

$C(A_0\ (\psi)) = \Big\{ \psi \in \Omega : |\psi_1(\theta)| \le \frac{4}{5(1-\beta_{11}-\gamma_{11})}$ and $|\psi_2(\theta)| \le \frac{3}{5(1-\beta_{22}-\gamma_{22})}, -h \le \theta \le 0 \Big\}$

is an attractor of (4.1) - (4.10) and the biggest set $\mathcal{S} = \Big\{ \varphi \in C : |\varphi_1(t_0) - \gamma_1 \varphi_1(t_0-h) - \gamma_2 \varphi_2(t_0-h)| \le \lambda \frac{1+\sqrt{5}}{2},$ $|\varphi_2(t_0)| \le \lambda \Big\}$ included in Ω is a positively invariant estimation of its domain of attraction.

5. CONCLUSION

This paper has provided two main results for estimating the behaviours of some nonlinear time-varying perturbed FDE of neutral type. The given conditions are independent-of-delay ones, and seem to be the first attempt to provide an easy-to-check procedure for the estimation of attractors. In what concerns the stability domains estimation, the method is rather classical in its principle, but the original point is that the results are directly workable. The inconvenient of these results is that the comparison system has to be of ODE type: thus, the considered neutral systems must have some restrictive characteristics, that correspond to assumption (2.3). This will be relaxed in further work by using a retarded-type (FDE) comparison system.

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