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NEW STABILITY CRITERIA FOR NONLINEAR NEUTRAL SYSTEMS

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Abstract

This paper gives sufficient stability conditions for nonlinear, time-varying, differential equations of neutral type. Considering particular vector Lyapunov functions (VLF), it proposes a systematic way of computing comparison systems which are of the retarded type. In the linear time-invariant case, very simple stability conditions are deduced.

1. Introduction

Functional Differential Equation (FDE) of the neutral type are often used to model engineering systems (see for instance [4], [8], [20]). As for many models, checking their stability properties constitutes a critical and complex step in the control design.

Stability of ordinary differential equations (ODE) has been widely studied (and is still), as well as for FDE of the retarded type (see the numerous references in [2], [7], [19]). However, criteria for neutral systems have not yet reached a comparable applicability and integration level.

In what concerns the stability of nonlinear FDEs, the most useful method is still the direct Lyapunov's one, which is based on the construction of some suitable function (Razumikhin, [13]) or functional (Krasovskii, [9]). This elegant method is known to be difficult in practice because of the complexity to construct such suitable function or functional, despite the two stage methods developped recently [6]. For many systems such a difficulty may be passed round by considering a simpler "comparison system" whose stability implies the stability of the original system. This comparison principle, at the outset applied for stability analysis of ODE [1][10][12], was further enlarged to *retarded* FDE [10], and more recently, a systematic result based on particular vector Lyapunov functions was proposed in [2], [3]. Concerning nonlinear *neutral* systems, a scalar comparison result was given in [8], [18], that the authors generalized to vector ones [15], [17]: but, in all this approaches of neutral systems, the obtained comparison systems are of the ODE type, which needs quite restrictive assumptions. This paper relaxes the major part of these constraining assumptions by defining comparison systems of the retarded type.

Here we consider the stability of the zero solution of the following FDE :

$$\dot{x}(t) = A_0(t, x(t), x(t-h))x(t)+ A_1(t, x(t), x(t-h))x(t-h)+B\dot{x}(t-h), \quad t \geq t_0, \tag{1.1}$$

with initial conditions, $x_{t_0} = \varphi$, $\varphi \in C^1([-h, 0] ; R^n)$, where

- $h > 0$ is a constant delay ;
- notation $M(.)$ and $M_k(.)$ stands for $M(t, x(t), x(t-h))$ and $M(t-kh, x(t-kh), x(t-(k+1)h))$ respectively where k is an integer;
- entries of $A_0(.)$ and $A_1(.)$ are continuous functionals ;
- B is a constant n×n matrix ;
- $x(t, t_0, \varphi) \in R^n$, also abridged as $x(t)$, is the solution of (1.1) with initial condition $\varphi = x_{t_0}$;
- x_t is the state function, $x_t(\theta) = x(t+\theta)\ \forall\ \theta \in [-h, 0]$.

The paper is organized as follows : section 2 presents assumptions and definitions. Section 3 gives the way to compute vector (corollary 3.1) or scalar (corollary 3.2) comparison systems, leading to general stability conditions (theorems 3.1 and 3.2). Then, the particular case of linear systems is considered (corollary. 3.3). In section 4, some examples illustrate the efficiency of the proposed results.

2. Notations, Assumptions and Definitions

Notations

- R^n is the real n-dimensional linear vector space with a norm $\|.\|$;

- $C = C^1([-h, 0] ; R^n)$ is the set of all differentiable functions mapping $[-h, 0]$ into R^n, with norm $\| \|_C$ defined by $\forall \in C$, $\|\varphi\|_C = \sup\{\|\varphi(\theta)\| : \theta \in [-h, 0]\}$;
- Ω is some subset of C , including a neighborhood of zero ;
- I_n is n×n idendity matrix ;
- if $\|u\|$ is a scalar norm of a vector u, the associated matrix norm of a matrix M(.) is defined by
  $\|M(.)\| = \sup_{x:\|x\|=1} \|M(.)x\|$;
  and the logarithmic norm of M(.) is (as in [7])
  $\gamma(M(.)) = \lim_{\Delta \to 0^+} \frac{1}{\Delta}(\|I_n + \Delta M(.)\| - 1)$;
- the vector x is partitionned into
  $x = [x_1^T, .., x_i^T, .., x_r^T]^T$ where $x_i \in R^{n_i}$, $\sum_{i=1}^{r} n_i = n$;
- $V : R^n \to R_+^r$ $(r \le n)$ is a regular vector Lyapunov function (RVLF) which components V_i are scalar norms of x_i ; according to the partition of x, matrix A is partionned into
  $$A = \begin{pmatrix} A_{11} . A_{1i} . A_{1r} \\ . \; . \; . \; . \; . \\ A_{i1} . A_{ii} . A_{ir} \\ . \; . \; . \; . \; . \\ A_{r1} . A_{ri} . A_{rr} \end{pmatrix},$$
- the r×r matrices V(A) and Γ(A) are defined as
  $V(A) = [V(A)_{ij}]$ with $V(A)_{ii} = V_i(A_{ii})$ and
  $V(A)_{ij} = \sup_{x_j : V_j(x_j) \neq 0} \{\frac{V_i(A_{ij}x_j)}{V_j(x_j)}\}, i \neq j$
  $\Gamma(A) = [\Gamma(A)_{ij}]$, $\Gamma(A)_{ii} = \gamma_i(A_{ii})$, $\Gamma(A)_{ij} = V(A)_{ij}$ $\forall i \neq j$;
- $\sup\{M(.)\}$ means the matrix $[m(.)_{ij}]$, $m_{ij}(.) = \sup M(.)_{ij}$, $t \ge t_0$, $x_t \in \Omega$;
- $\lambda(M)$ is an eigenvalue of matrix M, sp(M) the set of all eigenvalues of M , $\sigma(M) = \sup\{|\lambda(M)| : \lambda(M) \in sp(M)\}$;
- inequalities, absolute values, moduli for vector or matrices are to be considered for each corresponding entry.

Assumptions:

- A1) There exists an unique solution of (1.1) for any differentiable initial conditions (see [4] for this theory) ;
- A2) B is not nilpotent, that is there is no integer n such that $B^n \equiv 0$ (in the contrary case, system (1.1) could be reduced to a simpler retarded system).

Definition 2 [15] :
A dynamic system (A) is said to be a Comparison System of a dynamic system (B) with regard to the property $\mathcal{P}$ (for example, stability of its zero solution), if the verification of property $\mathcal{P}$ for system (A) implies the same property for system (B).

Definition 3 :
Let g be a functional mapping $R \times R^r \times C(]-\infty, 0] ; R^r)$ into R^r. The system

$$\dot{z}(t) = g(t, z(t), z_t), \quad z_{t_0} = \psi \qquad (2.1)$$

with solution $z(t) = z(t, t_0, \psi) \in R^r$, is an overvaluing system of (1.1) with respect to the RVLF V and set Ω, that this
$V(\varphi(\theta)) \le \psi(\theta), \forall \theta \le 0 \Rightarrow V(x(t)) \le z(t, t_0, \psi), \forall t \ge t_0$,
as soon as x_t takes its values in Ω if g is quasi -monotone nondecreasing in its second argument and nondecreasing in its third one [10][12].
The overvaluing system is said to be global if $\Omega = C$.

Remark1: An overvaluing system constitutes a comparison system if x(t) remains in Ω for all the time. Indeed, if $V(x(t)) \le z(t, t_0, V(\varphi))$ $\forall t \ge t_0$, then any property relative to stability verified by z(t) is verified by x(t).

3. Main results

Before we prove the main results let us state the following preliminary result in Lemma 3.1

Lemma 3.1
For any matrix A, $B \in R^{n \times n}$ and vector $x \in R^n$, the following inequality holds :
$V(ABx) \le V(A)V(B)V(x)$.

Proof : This immediately follows from $\forall A \in R^{n \times n}$, and $\forall x \in R^n$, $V(Ax) \le V(A)V(x)$.

Lemma 3.2 :
Consider a RVLF V and a subset Ω of C.
Suppose the following conditions
i) $\sigma(V(B)) < 1$,
ii) solution of (1.1) remains in Ω and is such that,

1) there exists a matrix $\Pi(.)$ with positive bounded coefficients such that
$\sum_{k=1}^{\infty} V(B)^{k-1} V((A_1(.) + BA_0(.))_k) \le \Pi(.)$, $t \ge t_0$,
2) solution of (3.1) exists and is unique.

Then the system (3.1)

$$\dot{z}(t) = \Gamma(A_0(.))z(t) + \sum_{k=1}^{\infty} V(B)^{k-1} V((A_1(.) + BA_0(.))_k) z(t-kh))$$
$$z(t) \in R^r, z_{t_0}(\theta) = \psi(\theta), \forall \theta \le 0 \qquad (3.1)$$

is a comparison system of (1.1) with regard to (asymptotic) stability with respect to VLF V and set Ω.

Proof : see appendix A.

The following corollary gives a simplification of lemma 3.2

Corollary 3.1
Suppose that the two conditions of lemma 3.2 hold for matrices $\Gamma^*(.)$ and $A_k(.)$ such that $\Gamma^*(.) \ge \Gamma(A(.))$ and $A_k(.) \ge V((A_1(.) + BA_0(.))_k)$, for $t \ge t_0$, $x_t \in \Omega$.
Then any system (3.2)

$$\dot{z}(t) = \Gamma^*(.)z(t) + \sum_{k=1}^{\infty} V(B)^{k-1} A_k(.) z(t-kh)),$$
$$z(t) \in R^r, z_{t_0}(\theta) = \psi(\theta), \forall \theta \le 0 \qquad (3.2)$$

is also a comparison system of (1.1) with regard to (asymptotic) stability with respect to VLF V and Ω.

The previous results have their equivalent in the scalar case. expressed in the following corollary.

Corollary 3.2
Consider norm $\|.\|$ and a subset Ω of C.
Suppose the following conditions
i) $\|B\| < 1$,
ii) solution of (1.1) remains in Ω and is such that,
1) there exists $\eta(.) < \infty$ such that
$\sum_{k=1}^{\infty}\|B\|^{k-1}a_k(.) \leq \eta(.)$, $t \geq t_0$, for any $\alpha(.)$ and $a_k(.)$ verifying $\gamma(A_0(.)) \leq \alpha(.)$, $\|(A_1(.) + BA_0(.))_k\| \leq a_k(.)$, for $t \geq t_0$, $x_t \in \Omega$
2) solution of (3.3) exists and is unique.
Then the system (3.3) defined by

$$\dot{z}(t) = \alpha(.)z(t) + \sum_{k=1}^{\infty}\|B\|^{k-1}a_k(.)z(t-kh), \qquad (3.3)$$

is a scalar comparison system of (1.1) with regard to (asymptotic) stability with respect to scalar norm $\|\ \|$ and Ω.

Now we can give stability conditions of zero solution of (1.1).

Theorem 3.1:
Suppose conditions of lemma 3.2 hold.
Then, the zero solution of (1.1) is stable (respectively asymptotically stable) if
i) there exists a positive constant vector u such that
$[\Gamma(A_0(.)) + \Pi(.)]u \leq 0$, $\forall\, t \geq t_0$, $x_t \in \Omega$,
ii) respectively there exists a positive constant vector u and scalar $\varepsilon > 0$ such that
$[\Gamma(A_0(.)) + \Pi(.)]u \leq -\varepsilon u$, $\forall\, t \geq t_0$, $x_t \in \Omega$.

Proof : see appendix B.

If V is a scalar norm we obtain Theorem 3.2

Theorem 3.2 :
Suppose conditions of corollary 3.2 hold.
Then, the zero solution of (1.1) is stable (respectively asymptotically stable) if
i) $\alpha(.) + \eta(.) \leq 0$, $\forall\, t \geq t_0$, $x_t \in \Omega$,
ii) respectively there exists a scalar $\varepsilon > 0$ such that
$\alpha(.) + \eta(.) \leq -\varepsilon$, $\forall\, t \geq t_0$, $x_t \in \Omega$.

In linear time invariant case, the two previous theorems are reduced to very easy-to-check criteria that we will summary in the following corollary 3 .2.

Corollary 3 .3
Suppose (1.1) is linear time invariant and consider scalar norm $\|.\|$ and a RVLF V.
1) If V(B) has all its eigenvalues inside the unit circle and $\Gamma(A_0)) + (I_r - V(B))^{-1}V(A_1 + BA_0)$ is the opposite of a M-matrix, then zero solution of (1.1) is aymptotically stable.
2) If $\|B\| < 1$ and $\gamma(A_0) + \dfrac{\|A_1 + BA_0\|}{1-\|B\|} < 0$,
then the zero solution of (1.1) is asymptotically stable.

4. Examples

4.1 Example 1

Let us consider the following system,

$$\dot{x}(t) = \begin{pmatrix} -2 & f_1(t,x(t),x(t-h)) \\ 0.5f_2(t,x(t),x(t-h)) & -2 \end{pmatrix} x(t) + \begin{pmatrix} 0 & 0.5 \\ 0 & 0 \end{pmatrix} x(t-h) + \begin{pmatrix} 0 & 0 \\ 0 & c \end{pmatrix} \dot{x}(t-h) \qquad (4.1)$$

where functions f_1 and f_2 are bounded, for $t \geq t_0$-h, as follows :

$|f_i(t, x(t), x(t-h))| \leq f_i$.

In the particular, linear case $f_1 = 0$, $f_2 = 1$, this example was considered in [5] and, further, in [16]. In our case, the functions f_i can be interpreted as bounded uncertainties on the model.
The problem, as in previous references, is to determine the values of parameter c ensuring the stability of the zero solution.
Linear case $f_1 = 0$, $f_2 = 1$:
In [16], by applying complex Lyapunov function, the asymptotic stability was proved for $-0.9848 < c < 0.9837$.
The result in [5] was more constraining : $0.25 < |c| < 0.52$.
Applying Corollary 3.3, a very simple calculation provides the condition $|c| \leq 0.468$. Our condition is more constraining than [5], however the calculation is immediate and our condition is still valid in the nonlinear case, as we will see now.
Nonlinear case $|f_i(.)| \leq f_i$:
Let us compute the parameters of a comparison system of (4.1). We have

$$\acute{A}_1 + BA_0(.) = \begin{pmatrix} 0 & 0.5 \\ 0.5cf_2(.) & -2c \end{pmatrix} \qquad (4.2)$$

Considering the simplest RVLF $V(x) = [|x_1|, |x_2|]^T$, the parameters involved in (3.2) can be expressed as

$$\Gamma^*(.) + \Pi(.) = \begin{pmatrix} -2 & 0.5+f_1 \\ \dfrac{0.5f_2}{1-|c|} & \dfrac{-2+4|c|}{1-|c|} \end{pmatrix}$$

This matrix is constant so it is sufficient to verify that $\Gamma^*(.) + \Pi(.)$ is the opposite a M-matrix.
The matrix $\Gamma^*(.) + \Pi(.)$ is the opposite of a M-matrix if

$$|c| < \frac{4-0.5f_2(0.5+f_1)}{8} . \qquad (4.3)$$

This is a condition for the asymptotic stability of the zero solution of (4.1). When f_1=0; f_2=1 we obtain previous results.
Note that this form of system can neither be studied by methods given in [5], nor [16].

4.2 Example 2

Consider a shunted power transmission line, described in [20] by the equation ($t \geq 0$, $x \in R$, $|b| < 1$, $g(0) = 0$),

$$\dot{x}(t) = -g(x(t)) + b\dot{x}(t-h), . \qquad (4.4)$$

It was shown in [20] that the zero solution of (4.4) is stable if $xg(x) > 0$ and $|b| < \frac{1}{2}$.

Using the "two-stage method", [6] showed that the zero solution of (4.4) is asymptotically stable if $xg(x) > 0$ and $|b| < 1$.

Let us derive another form of asymptotic stability condition of (4.4) in the case where the function g is differentiable with respect to x. In this case let define the function f(x) as the derivative of g with respect to x. Then note that (4.4) can be rewriten as

$$\dot{x}(t) = -\int_0^1 f(sx(t))ds\,x(t) + b\dot{x}(t-h),\quad t \geq 0.$$

Applying previous results we obtain the following condition of asymptotic stability :

if $\int_0^1 f(sx(t))ds \geq c > 0$, $t \geq 0$, then the zero solution of (4.4) is asymptotically stable if $|b| < \frac{1}{2}$.

In the linear case, that is $g(x(t)) = a(t)x(t)$, the asymptotic stability condition is reduced to $a(t) \geq c > 0$ and $|b| < \frac{1}{2}$; what is one condition given in [6] using two stages method as an asymptotic stability condition of zero solution of linear system $\dot{x}(t) = -a(t)x(t) + b\dot{x}(t-h)$.

4.3 Example 3

Consider now the gradient type equation. For example, if $\nabla G(x(t))$ denotes the gradient of G at x(t):

$$\dot{x}(t) = -\nabla G(x(t)) + B\dot{x}(t-h),\ t \geq 0,\ x \in R^n \qquad (4.5)$$

where $G: R^n \to R$ is two times continuously differentiable, with $G(0) = 0$, $\nabla G(0) = 0$.

This form of equation was also considered in [6] without assumptions that G is differentiable and $\nabla G(0) = 0$. It was shown that if $G(x) > 0$, $x \neq 0$ and $\|B\| < 1$ then zero solution of (4.5) is asymptotically stable.

Proceding as previously we obtain the following corollary.

Corollary 4.1:

The zero solution of (4.5) is asymptotically stable if one of the two following conditions holds :

1) $\|B\| < 1$ and there exists a scalar $\varepsilon > 0$ such that

$$\int_0^1 \gamma(-\nabla^2 G(sx(t)))ds + \sum_{k=1}^{\infty} \|B\|^{k-1} \int_0^1 \|B\nabla^2 G(sx(t-kh))\|ds \leq -\varepsilon .$$

2) V(B) has all its eigenvalues inside the unit circle and there exists $\varepsilon > 0$ and a r-vector $u > 0$ such that:

$$[\int_0^1 \Gamma(-\nabla^2 G(sx(t)))ds + \sum_{k=1}^{\infty} V(B)^{k-1} \int_0^1 V(B\nabla^2 G(sx(t-kh)))ds]u \leq -\varepsilon u.$$

For instance, take $G(x(t)) = x^T(t)Qx(t)$ where Q is a n×n constant matrix. Then $\nabla^2 G(x(t)) = Q + Q^T$ and the asymptotic stability condition is

1) $\|B\| < 1$ and $\gamma(-(Q + Q^T)) + \frac{\|B(Q + Q^T)\|}{1-\|B\|} < 0.$

or

2) V(B) has all its eigenvalues inside the unit circle and $\Gamma(-(Q + Q^T)) + (I_r - V(B))^{-1}V(B(Q + Q^T))$ is the opposite of a M-matrix.

Suppose Q has opposite off-diagonal entries, so that $Q + Q^T = \mathrm{diag}(-a_i)$, $a_i > 0$, $i=1, n$ and denote $\alpha = \min(a_i)$, $\beta = \max(a_i)$, $i=1,n$. Then the first condition of Corollary 4.1 implies that the zero solution of (4.5) is asymptotically stable if

$$\|B\| < \frac{\alpha}{\alpha+\beta} \leq \frac{1}{2}.$$

5. Conclusion

The present work provides contributions to the analysis of neutral type FDEs by giving scalar and vector asymptotic stability conditions. The main contribution of these results is relative to the computational simplicity, in the linear case as well as in the nonlinear case. In the linear time-invariant systems, one can consider these results as a generalization of [11] to neutral systems.

The given stability conditions are sufficient and independent of the size of delay.

Note also that conditions given in this paper are robust with respect to the perturbations of the coefficients of the system because they are based on majorations. In particular, these resuts can also be extended to the case where the matrix B is not constant, provided that its absolute value can be upperbounded by a constant one.

Appendix : Proofs

We will proove the results only in the vector case ; scalar case obsviously follows.

Appendix A : Proof of Lemma 3.3

Let function y(t) be defined by

$$y(t) = A_1(.)x(t-h) + B\dot{x}(t-h) \qquad \text{A1}$$

then (1.1) can be develped as

$$\dot{x}(t) = A_0(.)x(t) + \sum_{k=1}^{n} B^{k-1}(A_1(.) + BA_0(.))_k x(t-kh) + B^n y(t-nh) \qquad \text{A2}$$

where the matrix $(A_1(.) + BA_0(.))_k$ is defined as

$$y(t-(k-1)h) = (A_1(.) + BA_0(.))_k x(t-kh) + B\dot{x}(t-kh) \qquad \text{A3}$$

For some i, $1 \leq i \leq r$, consider the function Q_i defined in [7] as

$$Q_i[x_i,y_i] = \lim_{\Delta\to 0^+} \frac{1}{\Delta}[V_i(x_i + \Delta y_i) - V_i(x_i)].$$

Then

$$D^+V_i(x_i(t)) = Q_i[x_i(t), \dot{x}_i(t)]$$
$$= Q_i[x_i(t), \sum_{j=1}^{r} A_{0ij}x_j(t) +$$
$$\sum_{j=1}^{r} (\sum_{k=1}^{n} B^{k-1}((A_1 (.)+ BA_0(.))_k x(t-kh))_j + (B^n y(t-nh))_i] \quad A4$$

Considering the definition of Q_i we obtain the inequality

$$D^+V_i(x_i(t)) \le Q_i[x_i(t), A_{0ii}(.)x_i(t)] + \sum_{j=1, j\neq i}^{r} V_j(A_0(.)_{ij}x_j(t))$$
$$+ \sum_{j=1}^{r} V_j(\sum_{k=1}^{n} B^{k-1}(A_1 (.)+ BA_0(.))_k(t-kh)_j) + V_i(B^n z_i(t-nh)) \quad A5$$

by the definition of γ_i [7] yields

$$D^+V_i(x_i(t)) \le \gamma_i(A_{0ii}(.))V_i(x_i(t)) + \sum_{j=1, j\neq i}^{r} \Gamma(A_0(.))_{ij} V_j(x_j(t))$$
$$+ \sum_{j=1}^{r} V_j(\sum_{k=1}^{n} B^{k-1}(A_1 (.)+ BA_0(.))_k x(t-kh)_j) + V_i(B^n y_i(t-nh)) \quad A6$$

Doing this for i= 1 to r we obtain inequality :

$$D^+V(x(t)) \le \Gamma(A_0(.))V(x(t)) +$$
$$V(\sum_{k=1}^{n} B^{k-1}(A_1 (.)+ BA_0(.))_k x(t-kh)) + V(B^n y(t-nh)) \quad A7$$

or

$$D^+V(x(t)) \le \Gamma(A_0(.))V(x(t)) +$$
$$\sum_{k=1}^{n} V(B^{k-1}(A_1 (.)+ BA_0(.))_k)x(t-kh)) + V(B^n y(t-nh)) \quad A8$$

and applying lemma 3.1 finaly yields

$$D^+V(x(t)) \le \Gamma(A_0(.))V(x(t)) +$$
$$\sum_{k=1}^{n} V(B)^{k-1} V((A_1 (.)+ BA_0(.))_k) V(x(t-kh))$$
$$+ V(B)^n V(z(t-nh)) \quad A9$$

According to condition $\sigma(V(B)) < 1$ and the fact that $\forall\, t \ge t_0$ there exists an integer n_0 such that $V(y(t-nh))$ is bounded by initial conditions for $n \ge n_0$, $V(B)^n V(y(t-nh)) \to 0$ when $n \to +\infty$ then it is sufficient that the sum

$$\Sigma(t) = \lim_{n\to+\infty} \Sigma_n(t)$$

where $\Sigma_n(t) = \sum_{k=1}^{n} V(B)^{k-1} V((A_1(.)+ BA_0(.))_k) V(x(t-kh))$ A10

exists for obtaining an inequality depending only on the component x so if we consider the r vector function m(t), defined by

$$m_i(t) = \sup_{s\le t}\{V_i(x_i(s))\} \quad A11$$

we have the inequality

$$\Sigma_n(t) \le [\sum_{k=1}^{n} V(B)^{k-1} V((A_1 (.)+ BA_0(.))_k)] m(t) \quad A12$$

and as the sum $\sum_{k=1}^{n} V(B)^{k-1} V((A_1 (.)+ BA_0(.))_k$ exists according to condition ii) and this imply the existence of $\Sigma(t)$ and finaly the inequality

$$D^+V(x(t)) \le \Gamma(A_0(.))V(x(t))$$
$$+ \sum_{k=1}^{\infty} V(B)^{k-1} V((A_1(.) + BA_0(.))) V(x(t-kh)) \quad A13$$

as $n \to +\infty$; the off diagonal coefficients of matrix $\Gamma(A_0(.))$ and all coefficients of matrix $V(A_1(.)+BA_0(.))_k))$ are nonnegative, so functional g defined by

$$g(t, z(t), z_t) = \Gamma(A(.))\, z(t) +$$
$$\sum_{k=1}^{\infty} V(B)^{k-1} V(A_1 (.)+ BA_0(.))_k) z(t-kh) \quad A14$$

which appears in inequality (A13) is quasi-monotone nondecreasing in its second argument and nondecreasing in its third argument so by definition 2 and remark 1 conclusion yields.

Appendix B : Proof of Theorem 3.2

Before prooving theorem 3.3 let us give without proof this lemma

Lemma B1 [7][10] (comparaison principle)
If $\Gamma(A_0(.))$ has all its off-diagonal coefficients nonnegative and given two vector function $z_1(t)$, $z_2(t) \in R_+^r$ verifying differential inequality

$$D^+z_1(t) \le \Gamma(A_0(.))\, z_1(t)$$
$$+ \sum_{k=1}^{\infty} V(B)^{k-1} V(A_1 (.)+ BA_0(.))_k) z_1(t-kh)) \quad B1$$

$$\dot{z}_2(t) \ge \Gamma(A_0(.))\, z_2(t)$$
$$+ \sum_{k=1}^{\infty} V(B)^{k-1} V(A_1 (.)+ BA_0(.))_k) z_2(t-kh)) \quad B2$$

if $z_1(t_0+\theta) \le z_2(t_0+\theta)$, $\theta \le 0$ then $z_1(t) \le z_2(t)$ for all $t \ge t_0$

1) stability : Consider the set $\{\varphi : V(\varphi) < \gamma\}$, where γ is a r-vector with positive components. Then it is sufficient to show that this set is a positive invariant set for system (1.1). Let suppose the contrary, then there exist a fiist time t_1 and an index i such that $V_i(x(t_1)) = \gamma_i$. But acording to lemma 3.2 any solution of (1.1) verify inequality (A13). For any initial condition φ there exists a positive diagonale matrix Δ such that $V(\varphi) = \Delta u$. Define $z_1(t) \equiv V(x(t))$ and $z_2(t) \equiv V(\varphi) = \Delta u$. These functions satisfy inequalities (B1) and (B2) for $t_0 \le t \le t_1$ and so, according to lemma B1
$z_1(t) \le z_2(t) \equiv V(\varphi) < \gamma$ for $t_0 \le t \le t_1$ which contradicts the definition of t_1; so, for any γ, the set $\{\varphi : V(\varphi) < \gamma\}$ is positively invariant for (1.1) which means the stability of zero solution of (1.1).

2) asymptotic stability : I is sufficient to proove that zero solution of (3.1) is asymptotically stable. Consider the system (3.1) and define the function v(z(t)) as:

$$v(z(t)) = \max_{1\le i\le r} \{\frac{|z_i(t)|}{u_i}\} \quad B4$$

For each t there exists an indice p such that $v(z(t)) = \frac{|z_p|}{u_p}$ and its derivative verifies

$$\frac{d^+v(z(t))}{dt} \leq \frac{1}{u_p} [\gamma_p(A_0(.)_{pp})|z_p| + \sum_{j=1, j\neq p}^{r} \Gamma(A(.))_{pj})z_j(t)| + \sum_{k=1}^{\infty}\sum_{j=1}^{r}[V(B)^{k-1}V(A_1+BA_0)_k))]_{pj}|z_j(t-kh))|] \qquad B5$$

or

$$\frac{d^+v(z(t))}{dt} \leq \frac{1}{u_p}[\gamma_p(A_0(.)_{pp})|z_p(t)| + \sum_{j=1, j\neq p}^{r} \Gamma(A(.))_{pj}|z_j(t)| + \sum_{k=1}^{\infty}\sum_{j=1}^{r}[V(B)^{k-1}V(A_1+ BA_0)_k))]_{pj} |z_j(t))|] \qquad B6$$

along the trajectories verifying $|z_j(t-kh)| \leq |z_j(t)$.
Then, according to definition of $v(z(t))$, we have

$$\frac{d^+v(z(t))}{dt} \leq \frac{1}{u_p} [[\Gamma(A_0(.))u]_p + \sum_{k=1}^{\infty}\sum_{j=1}^{r}[V(B)^{k-1}V(A_1(.)+BA_0(.))_k)]_{pj}u_j v(z(t))] \qquad B7$$

or $\frac{d^+v(z(t))}{dt} \leq \frac{1}{u_p}[(\Gamma(A_0(.))+\Pi(.))u]_p v(z(t)) < -\varepsilon v(z(t))$

so $v(z(t))$ is a Lyapunov-Razumikhin function, and the conclusion yields.

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