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STABILITY OF NEUTRAL SYSTEMS VIA VECTOR-LYAPUNOV FUNCTION

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Abstract

This paper provides a stability criterion for nonlinear neutral time-delay systems. It defines the notion of Degenerate Comparison System, which is an extension of a previous Vector-Lyapunov Functions approach available for time-delayed systems, and of a scalar-type result relative to neutral ones. From this, the obtained stability co nditions are to be checked on two simple systems : an Ordinary Differential Equation (ODE), and an Ordinary Difference Equation.

1. INTRODUCTION

The direct Lyapunov's method remains of basic interest for studying stability of nonlinear Functional Differential Equations (FDE). It refers either to the Razumikhin's approach [13] using ordinary *functions*, or to Krasovskii's approach [9] using *functionals* that are defined along the trajectories of the system. Despite of recent results [8] the general construction of such function(al)s remains difficult in many cases.

An other approach is based on differential inequalities : many authors obtained criteria based on a so-called *comparison system*, that is expected to be more easy to study, and which stability implies the stability of the original system. This method was first applied for investigating stability of ODE (see for example [11][12]), and further, for FDE (Lakshmikantham [10][3]). Recently, the authors [1][2] proposed a systematic method for constructing vector-comparison systems for FDE of retarded type, that was shown to provide less conservative results than scalar-comparison-based criteria. However such a vectorial method was not yet applied for FDE of neutral type, and the only existing approach for this neutral systems was based on scalar comparison systems [5][6], with the interesting concept of *degenerate* Lyapunov functional.

This present work enlarges this concept to Vector Lyapunov Function (VLF), in order to construct a vector-comparison system.

A first part presents a systematic way of constructing a Degenerate Comparison System (DCS), and derives sufficient stability conditions.

A second part applies these results to an example.

Notations and assumptions

The considered systems are of neutral type, described by :

$$\frac{dDx_t}{dt} = A(t,x_t)Dx_t + F_2(t, x_t) \qquad (1.1)$$

$$Dx_t = x(t) - Bx(t-h)$$

where :

- h is a positive constant representing the time-delay
- B is a matrix with constant coefficients ;
- $C = C([t_0\text{-}h, t_0], R^n)$ is the set of all continuous functions mapping $[t_0\text{-}h, t_0]$ onto R^n,
- $x(t) \in R^n$ and $x_t \in C$ is the state function classically defined by $\forall\, \theta \in [t_0\text{-}h, t_0]$, $x_t(\theta) = x(t+\theta)$;
- $A(t,x_t) \in R^{n \times n}$ and F_2 can involve unknown coefficients ; they are assumed to present sufficient smoothness properties ensuring the existence of a solution of (1.1) (see for example [4]). $A(t,x_t)$ is bounded for bounded x_t, and in the following A(.) represents a simplified notation for $A(t,x_t)$.

The solution $x_t \equiv 0$ is an equilibrium of (1.1), and this is guaranted by :

$$F_2(t,0) = 0,\ \forall\, t \in R \qquad (1.2)$$

R^n is decomposed into the direct sum of r subspaces R^{n_i} (i = 1 to r), with $x_i \in R^{n_i}$ the projection of x onto R^{n_i}.

- V is some candidate Vector Lyapunov Function (VLF), where V_i is a scalar norm on R^{n_i}

$$V : R^n \to R^r\ (r \leq n),$$

$$V(x) = [V_1(x_1),..,V_i(x_i),..,V_r(x_r)]^T \qquad (1.3)$$

(V is said to be a Regular Vector Norm).

- $\gamma_i(S)$ is the measure, or logarithmic norm [7][8], of the square matrix $S \in R^{n_i \times n_i}$ with respect to V_i.

It is assumed that F_2 verifies a boundedness-type condition on the subset Ω :

$$V(F_2(t,x_t)) \le N(t)V(Dx_t)$$
$$\forall\, x_t \in \Omega \subseteq C \qquad (1.4)$$

where N(t) is a matrix of size r with scalar continuous nonnegative coefficients.

Lastly, $|\chi|$ denotes the absolute value of any scalar χ

2. COMPARISON SYSTEMS

Definition 1 :

A dynamic system (A) is said to be a *Comparison System* of a dynamic system (B) with regard to (asymptotic) stability, if the (asymptotic) stability of the zero solution of (A) implies the (asymptotic) stability of the zero solution of (B).

Definition 2 :

Let $g(t,.)\ R \times R^r \rightarrow R^r$ be a quasi-monotone nondecreasing function with regard to its second argument, this is, verifying the usual Ważewski conditions [12][14].

Then system (2.1) :

$$D^+y = g(t,y)\ \forall t \ge t_0,\ \forall\, y \in R^r \qquad (2.1)$$

is a *Degenerate Comparison System* (DCS) of (1.1) with respect to the VLF V and the set Ω, if the following inequality is satisfied along every motion of (1.1)

$$D^+V(Dx_t) \le g(t,V(Dx_t))\ \forall\, t \ge t_0,\ \forall\, x_t \in \Omega \qquad (2.2)$$

If $\Omega = C$, the degenerate comparison system is said to be global.

Remark :

In this paper, we shall only use functions g of linear time-varying type :

$$g(t,y(t)) = (M(t) + N(t))y(t) \qquad (2.3)$$

where $M(t) : R \rightarrow R^{r \times r}$ is a time-continuous matrix with nonnegative off-diagonal coefficients, corresponding to inequalities of the type :

$$D^+V(Dx_t) \le (M(t) + N(t))V(Dx_t)$$
$$\forall\, t \ge t_0,\ \forall\, x_t \in \Omega. \qquad (2.4)$$

Lemma 1 : construction of DCS

Any system of form (1.1) with assumption (1.4) admits a Degenerate Comparison System with respect to the VLF V *and the set* Ω*, defined by*

$$\frac{dy(t)}{dt} = (M(t) + N(t))y(t) \qquad (2.5)$$

where N(t) *is defined by (1.4) and* M(t) *is defined by*

$$M(t) = [\mu_{ij}(t)],\ \textit{with}$$
$$\mu_{ii}(t) = \sup\ \{\ \gamma_i(A(.)_{ii})\ : x_t \in \Omega\ \},$$
$$\mu_{ij}(t) = \sup\ \{\frac{V_i(A(.)_{ij}(D_t)_j)}{V_j((Dx_t)_j)} :\ x_t \in\ \Omega\ \}.$$

Proof : see appendix

corollary :

Any matrix $M_1(t) + N_1(t)$ *such that* $\mu_{1ij}(t) \ge \mu_{ij}(t)$, $\mu_{1ii}(t) = \mu_{ii}(t)$ *and* $\nu_{1ij}(t) \ge n_{ij}(t)$ $(N(t) = [\nu_{ij}(t)])$ *also defines a DCS of (1.1) with respect to the VLF* V *and the set* Ω.

This corollary is important because it allows to derive more simple DCS, that are easier to analyze (linear time-invariant ones, for example).

Lemma 2 : (additional conditions for a DCS to be a comparison system)

If B *has all its eigenvalues inside of the unit circle, then the system (2.5) is a comparison system of (1.1) with regard to stability and asymptotic stability.*

Proof :

Remark that the function $y \rightarrow (M(t) + N(t))y$ is quasi monotone nondecreasing with respect to y.

Let us call $y(t, V(D\varphi))$ a solution of (2.5) with initial condition $y(t_0) = V(D\varphi)$ then it is known [12][14] that :

$$V(Dxt) \le y(t,\ V(D\varphi))\ \forall\ t \ge t_0, \qquad (2.6)$$

Considering a norm $\|\ \|$ of R^n and $\|\ \|_i$ the norm induced by $\|\ \|$ on the subspace R^{n_i} (that is $\|\ x_i\ \|_i = \|\ [0, ..., 0, x_i, 0, ..., 0]^T\ \|$) then :

$$\|\ Dx_t\ \| \le \sum_{i=1}^{r} \|(Dx_t)_i\|_i \le \sum_{i=1}^{r} c_i V_i((Dx_t)_i)$$
$$\le \sum_{i=1}^{r} c_i y_i(t, V(D\varphi)) = k(t)$$

where the r coefficients c_i are defined by the classical "norm equivalence" relations :

$$\|y\|_i \le c_i V_i(y) \qquad (2.7)$$

Applying the result,given in [5][6] we obtain the conclusion.

Theorem (stability conditions)

Let us note $\Phi(t_0,t)$ *the transition matrix of system (2.5) .*

If all the eigenvalues of B *lie in the unit circle, and if there exists* $\eta < \infty$ *such that* $(\forall t \ge t_0)$ $(\ \|\Phi(t_0,t)\| \le \eta)$ *(respectively* $\|\Phi(t_0,t)\| \rightarrow 0$ *as* $t \rightarrow \infty$*), then the trivial zero solution of (1.1) is stable (respectively asymptotically stable).*

Proof : This is a direct application of previous lemmas and ODE stability conditions.

3. EXAMPLE

Let us consider the system with an unknown varying parameter $d(.)$ (with known bounds) and $\alpha(.)$ (with bounds to be calculated in order to ensure the stability) :

$$\frac{dDx_t}{dt} = \begin{pmatrix} -4+\cos(t) & \sin(x_1(t)) \\ -0.5d(.) & -5 \end{pmatrix} Dx_t + \begin{pmatrix} 0 \\ [1+d(.)]\alpha(.)x_2(t) \end{pmatrix} \quad (3.1)$$

$$Dx_t = x(t) - \begin{pmatrix} \gamma & 0 \\ 0 & 0 \end{pmatrix} x(t-h),\ x_{t0} = \varphi \quad (3.2)$$

$d(.)$ is a scalar time-continuous function verifying $|d(.)| \le 1$ for all $t \ge t_0$ and all x_t in C, and it is assumed that γ is a constant verifying $|\gamma| < 1$.
The problem is to look for conditions on $\alpha(.)$ such that the zero solution of (3.1) is asymptoticaly stable.

Let us consider the VLF defined by
$V(x) = [|x_1|,|x_2|]^T$ where x_i represents the i^{th} component of vector x. Along the motions of (3.4) the following inequality holds :

$$D^+V(Dx_t) \le \begin{pmatrix} -3 & 1 \\ 0.5 & -5 \end{pmatrix} V(Dxt) + \begin{pmatrix} 0 \\ 2|\alpha(.)|\ |x_2(t)| \end{pmatrix},$$

$\forall\ t \ge t_0$ and x_t in C

According to the definition of Dx_t we obtain the inequality :

$$D^+V(Dx_t) \le \begin{pmatrix} -3 & 1 \\ 0.5 & -5 \end{pmatrix} V(Dx_t) + \begin{pmatrix} 0 & 0 \\ 0 & 2|\alpha(.)| \end{pmatrix} V(Dx_t),$$

$\forall\ t \ge t_0$ and x_t in C

We obtain the matrix $M(t) + N(t)$ of DCS (2.5) as :

$$M(t) + N(t) = \begin{pmatrix} -3 & 1 \\ 0.5 & -5+2|\alpha(.)| \end{pmatrix}$$

A condition for the zero solution of the global DCS (2.5) to be asymptotically stable is :

$$|\alpha(.)| < \frac{29}{12},\ \forall\ t \ge t_0 \text{ and } x_t \text{ in C.} \quad (3.3)$$

And as $|\gamma| < 1$, applying lemma 2 proves that (3.3) is also a sufficient asymptotic stability condition of the zero solution of (3.1).

APPENDIX : proof of lemma 1

Let us introduce the function
$Q_i : R^{n_i} \times R^{n_i} \to R$ defined by :

$$Q_i[x_i, y_i] = \lim_{\Delta\to 0+} \frac{1}{\Delta}[V_i(x_i + \Delta y_i) - V_i(x_i)] \quad (A1)$$

The function Q_i is well defined for any scalar norm V_i on R^{n_i}. It is known [7] that for any continuously differentiable function $(Dx_t)_i$,

$$\frac{d^+V_i((Dx_t)_i)}{dt} = Q_i[(Dx_t)_i, \frac{d(Dx_t)_i}{dt}] = Q_i[(Dx_t)_i, \sum_{j=1}^{r} A(.)_{ij}(Dx_t)_j + F_{2i}(t,x_t))] \quad (A2)$$

by developing the sum and considering the definition of function Q_i, we obtain :

$$\frac{d^+V_i((Dx_t)_i)}{dt} \le Q_i[(Dx_t)_i, A(.)_{ii}(Dx_t)_i] + \sum_{j=1, j\ne i}^{r} V_i(A(.)_{ij}(Dxt)_j) + V_i(F_{2i}(t,x_t)) \quad (A3)$$

Let us define (for i=1...r) :

$$\mu_{ii}(t) = \sup \{\ \gamma_i(A(.)_{ii}) : x_t \in \Omega\} \quad (A4)$$

Remark that

$$\gamma_i(A_{ii}) = \sup_{x_i\ne 0} \{V_i(x_i)^{-1}Q_i(x_i, A_{ii}x_i)\} \quad (A5)$$

Then relations (A3) and (A5) imply :

$$\frac{d^+V_i((Dx_t)_i)}{dt} \le \gamma_i(A(.)_{ii}V_i((Dx_t)_i) + \sum_{j=1}^{r} V_i(A(.)_{ij}(Dx_t)_j) + V_i(F_{2i}(t,x_t)) \quad (A6)$$

$$\text{Let } \mu_{ij}(t) = \sup \{ \frac{V_i(A(.)_{ij}(Dx_t)_j)}{V_j((Dx_t)_j)} : x_t \in \Omega\ \} \quad (A7)$$

(A6) with notations (A4) and (A7) yields :

$$\frac{d^+V_i((Dx_t)_i)}{dt} \le \mu_{ii}(t)V_i((Dx_t)_i + \sum_{j=1, j\ne i}^{r} \mu_{ij}(t)V_j((Dx_t)_j) + V_i(F_{2i}(t,x_t)) \quad (A8)$$

$D^+V(Dx_t) \le M(t)V(Dx_t) + V(F_2(t,x_t))$
where the matrix $M(t)$ is defined by :
$M_{ii}(t) = \mu_{ii}(t)$; $M_{ij}(t) = \mu_{ij}(t)$, $i \ne j$
Considering the assumption (1.4) finally yields :

$$D^+V(Dx_t) \le (M(t) + N(t))V(Dx_t) \quad (A9)$$

It is obvious that the matrix $M(t) + N(t)$ has nonnegative off-diagonal elements, which proves lemma 1.

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