2.1 VECTOR LYAPUNOV FUNCTIONS: NONLINEAR, TIME-VARYING, ORDINARY AND FUNCTIONAL DIFFERENTIAL EQUATIONS*

P. BORNE, M. DAMBRINE, W. PERRUQUETTI
and J.P. RICHARD

Ecole Centrale de Lille, LAIL (CNRS), Lille, France

1 Introduction

When dealing with the qualitative analysis of solutions of complex large scale functional differential equations (or ordinary differential equations) the seminal results of Lyapunov can be used and, more precisely, the second Lyapunov method introducing Lyapunov functions. Unfortunately, the more complex the considered dynamical system, the more difficult it is to find a Lyapunov function.

When faced with such a complex problem, the following general remarks can be made:

R1) Some of the complex aspects can be reduced to simpler connected problems. For example, this principle is used for the study of nonlinear dynamical systems: the linearized system provides some knowledge of the original system (under well-known conditions).

R2) Following Descartes' precepts, it can be relevant to split the problem into several components in order to extract an easier understanding of the initial problem through the understanding of each obtained components.

According to these remarks, it is interesting to apply the approach which consists in transforming a complex, large-scale dynamical system into several more simple dynamical systems of reduced dimensions. Let us examine these remarks by applying them to two examples: the first example will illustrate R1) and the second one R2).

Advances in Stability Theory* (Ed.: A.A. Martynyuk). Stability and Control: Theory, Methods and Applications, Taylor & Francis, London, **13 (2003) 49–73.

Example 1.1 Consider the system

$$\frac{dx}{dt} = 2x(-2 + \sin(t) + x), \quad t \in \mathbb{R}, \quad x \in \mathbb{R}. \tag{1}$$

Introducing the variable $z = \mathrm{sign}\,(x)x$, this gives[1]

$$\frac{dz}{dt} = 2z(-2 + \sin(t) + x), \quad \text{if } x \neq 0, \tag{2}$$

$$\frac{dz}{dt} \leq 2z(-1 + z), \tag{3}$$

$$z(t) \leq \frac{z_0}{z_0 + (1 - z_0)\exp(2(t - t_0))}. \tag{4}$$

It can be deduced that the origin of system (1) is asymptotically stable and an estimate of its domain of attraction is $(-\infty, 1)$.

In this first example, a *comparison system* (CS) was implicitly used, since the solution of (3) has been upperbounded by the solution of the ordinary differential equation

$$\dot{y} = 2y(-1 + y).$$

Such a comparison system presents several interesting properties:

- Its solutions overestimate the actual system behaviour.
- It may infer a qualitative property $\mathcal{P}$ for the initial system, in this case the CS will be a *P-comparison system* (P-CS) with respect to the property $\mathcal{P}$. For instance, in Example 1.1, $\mathcal{P}$ is the exponential stability property.
- It may not depend on disturbances or time variable: this takes much of the hard work out of the study.
- It can be described by lower-order dynamical systems. In order to illustrate this last property, let us examine the following example:

Example 1.2 Consider the system

$$\frac{dx}{dt} = \begin{pmatrix} -2 + \sin(t) + \frac{1}{4}(x_1^2 + x_2^2) & -\sin(t) \\ \sin(t) & -2 + \sin(t) + \frac{1}{4}(x_1^2 + x_2^2) \end{pmatrix} x, \tag{5}$$

$$t \in \mathbb{R}, \quad x \in \mathbb{R}^2,$$

[1] As the derivative of the sign function at $x = 0$ has no meaning in the usual sense, we exclude this case in a first stage. In fact, using a more general notion of derivative (see [9]), a similar result can be directly obtained. Note that (4) is still valid for $x = 0$.

introducing the variable $v = \frac{1}{4}(x_1^2 + x_2^2)$, this gives

$$\frac{dv}{dt} = 2v(-2 + \sin(t) + v), \quad t \in \mathbb{R}, \quad v \in \mathbb{R}_+. \tag{6}$$

Then, using Example 1.1, one can conclude that the original system (5) is asymptotically stable and an estimate of its domain of attraction is $\{x \in \mathbb{R}^2 : (x_1^2 + x_2^2) < 4\}$.

However, this reduction of dimension may not be an advantage in all situations. In fact, using a standard Lyapunov function (v) always leads to a first order comparison system, which may represent a drastic cut-down. It thus seems interesting to reduce the loss of information by using not one but several Lyapunov functions: this leads to the notion of *vector Lyapunov function* (VLF) for which each component provides information about a part of the dynamics.

Finally, these introductory examples show that comparison techniques [2 – 4, 10, 25, 30 – 32, 37, 50] combined with VLF [1, 10, 20, 21, 23, 25, 32, 33, 37, 42] are certainly an interesting alternative for tackling the study of stability properties of dynamical systems.

Therefore the paper will be divided as follows:

- Section 2 briefly presents the notations used throughout the paper.
- Section 3 sets up the general framework for the considered dynamical systems, i.e. nonlinear, time-varying, ordinary and/or functional differential equations.
- Section 4 concerns the notion, the construction and the properties of CS. We shall see that, among VLF, the particular case of convex ones, illustrated by the *vector norms* (VN), provides a very convenient tool for the construction of CS.
- Section 5 presents different results concerning qualitative set properties for the considered dynamical systems: stability, attractivity, positive invariance. The definitions are recalled at the beginning of the section.
- Section 6 concerns the application of some results of the previous parts to the particular case of ordinary differential systems.

2 Notations

- $\mathcal{C} = C([-\tau, 0], \mathbb{R}^n)$, the set of continuous maps from $[-\tau, 0]$ into $\mathbb{R}^n$;
- $\mathcal{C}(\mathcal{D}) = C([-\tau, 0], \mathcal{D})$, the set of continuous maps from $[-\tau, 0]$ into $\mathcal{D}$;
- $\mathcal{C}_+ = \{y \in \mathcal{C} : y(s) \geq 0,\ s \in [-\tau, 0]\}$;
- x_t element of $\mathcal{C}$ associated with a map $x : \mathbb{R} \to \mathbb{R}^n$ by $x_t(s) = x(t + s)$, for all $s \in [-\tau, 0]$;
- φ_x element of $\mathcal{C}$, defined by $\varphi_x(s) = x$, for all $s \in [-\tau, 0]$;

- $B(x,\varepsilon)$ is an ε-ball centered at x in the metric space from which x is defined with the distance function $\rho(\cdot,\cdot)$;
- $\mu(A)$ is the matrix measure of the square matrix A;
- $a\mathcal{R}b$, elementwise relation $\mathcal{R}$ (a and b are vectors or matrices): for example $a<b$ (vectors) means $\forall i\colon a_i<b_i$.

3 Framework

A large number of processes can be modelled by a *functional differential equation:*

$$\dot{x}(t)=f(t,x(t),x_t,d), \tag{7}$$

$$x_{t_0}=\varphi\in\mathcal{C}, \tag{8}$$

where $t\in\mathbb{R}$ is the time variable, $d\in\mathcal{S}_d$ is either a vector or a function representing disturbances or parameter uncertainties of the system, $\mathcal{S}_d$ is a set of vector or functions for which some bounds are usually supposed to be known, $x(t)\in\mathbb{R}^n$ is a set of internal variables, x_t is the map defined by

$$\begin{aligned} x_t\colon [-\tau,0] &\to \mathbb{R}^n,\\ s &\mapsto x(t+s). \end{aligned} \tag{9}$$

In the paper, when the case $\tau=0$ is considered, we shall use $x_t \triangleq x(t)$ for the sake of simplicity. In this case, (7) can be directly rewritten as

$$\dot{x}(t)=f(t,x(t),d), \tag{10}$$

$$x_{t_0}=x(t_0)=\varphi\in\mathbb{R}^n. \tag{11}$$

This represents a slight misuse of terminology since, in (7), f is a functional, x_{t_0} is a function and in (10), f is a function, x_{t_0} is a vector.

Assumption 3.1 *It is assumed that system (7) has solutions (for example f satisfies Carathéodory conditions: see [26]) defined over a maximal interval denoted by $\mathcal{I}_{(7)}(t_0,\varphi)$ where t_0 is the initial time and φ is the initial function defined over $[-\tau,0]$ (most of the time φ is supposed to belong to $\mathcal{C}$).*

4 Comparison Systems

As seen in Example 1.1 of the introduction, it is interesting to obtain information about a complex system through a simpler one whose solutions overvalue the solutions of the initial system. Wazewski's contribution [50] is probably one of the most

important in this field: it concerns differential inequalities and gives necessary and sufficient hypotheses ensuring that the solution of $\dot{x}=f(t,x)$, with initial state x_0 at time t_0 and function f satisfying the inequality $f(t,x)\le g(t,x)$ is overvalued by the solution of the so-called "comparison system" $\dot{z}=g(t,z)$, with initial state $z_0\ge x_0$ at time t_0, or, in other words, conditions on function g that ensure $x(t)\le z(t)$ for $t\ge t_0$. These results were extended to many different classes of dynamical systems ([2, 10, 23, 30, 33, 48]).

In the next subsection, we first define the notions of *comparison system* (CS) and of *P-comparison system* (P-CS). The next subsection is then devoted to the construction of such CS.

4.1 Definitions

Focusing on the two systems:

$$\dot{x}(t)=f(t,x(t),x_t),\quad x(t)\in\mathbb{R}^n, \tag{12}$$

$$\dot{z}(t)=g(t,z(t),z_t),\quad z(t)\in\mathbb{R}^n, \tag{13}$$

we respectively note $z(t;t_0,\varphi_2)$ and $x(t;t_0,\varphi_1)$ the solutions of (13) with initial condition φ_2 and of (12) with initial condition φ_1.

Definition 4.1 System (13) is said to be a *comparison system* of (12) over $\Omega\subset\mathcal{C}$ if:

$$\forall\,(\varphi_1,\varphi_2)\in\Omega^2\colon\ \mathcal{I}=\mathcal{I}_{(12)}(t_0,\varphi_1)\cap\mathcal{I}_{(13)}(t_0,\varphi_2)\text{ is not reduced to }\{t_0\},$$
$$\varphi_2\ge\varphi_1\Longrightarrow z(t;t_0,\varphi_2)\ge x(t;t_0,\varphi_1),\quad \forall\,t\in\mathcal{I}.$$

Obviously, one can go beyond this concept to derive a qualitative analysis for positive solutions. For example, if $z(t;t_0,\varphi_2)\ge x(t;t_0,\varphi_1)\ge 0$ and if solution $z(t)$ converges to zero, so does $x(t)$. For this reason, we introduce the following notion:

Definition 4.2 System (13) is a *P-comparison system* of (12) for property $(\mathcal{P})$ if

$$[(\mathcal{P})\text{ holds for (13)}\Rightarrow(\mathcal{P})\text{ holds for (12)}].$$

Example 4.1 If we consider a nonlinear system

$$\dot{x}(t)=f(x(t)), \tag{14}$$

then a P-comparison system for the property of uniform asymptotic stability of the zero solution is given by the first order approximation

$$\dot{z}(t)=Az(t), \tag{15}$$

where $A=\left(\frac{\partial f}{\partial x}\right)_{x=0}$.

A question naturally arises concerning properties of the function g ensuring that (13) is a comparison system of (12) over Ω. For this, the following notion is required:

Definition 4.3 A *functional*

$$g\colon \mathbb{R}\times\mathbb{R}^n\times\mathcal{C}\to\mathbb{R}^n$$
$$(t,x,y)\mapsto g(t,x,y)$$

(1) is *quasi-monotone non-decreasing in x* iff:

$$\forall t\in\mathbb{R},\quad \forall y\in\mathcal{C},\quad \forall(x,x')\in\mathbb{R}^n\times\mathbb{R}^n: \tag{16}$$
$$\forall i\in\{1,\dots,n\}[(x_i=x_i')\wedge(x\le x') \Rightarrow g_i(t,x,y)\le g_i(t,x',y)], \tag{17}$$

(2) is *non-decreasing in y* iff:

$$\forall t\in\mathbb{R},\ \forall x\in\mathbb{R}^n,\ \forall(y,y')\in\mathcal{C}\times\mathcal{C}\colon [y\le y']\Rightarrow g(t,x,y)\le g(t,x,y'), \tag{18}$$

(3) is *mixed quasi-monotone non-decreasing in x, non-decreasing in y* iff:

$$\forall t\in\mathbb{R},\quad \forall(x,x')\in\mathbb{R}^n\times\mathbb{R}^n,\quad \forall(y,y')\in\mathcal{C}\times\mathcal{C}: \tag{19}$$
$$\forall i\in\{1,\dots,n\}[(x_i=x_i')\wedge(x\le x')\wedge(y\le y') \Rightarrow g_i(t,x,y)\le g_i(t,x',y')]. \tag{20}$$

Remark 4.1 The latter definition is a special case of mixed quasimonotonicity given in [30]. More general versions also exist (see [2, 3, 25, 32, 33]) and additional conditions are sometimes given (see [25, 23], or [50]).

The following result may easily be proved:

Lemma 4.1 *A functional $g\colon (t,x,y)\mapsto g(t,x,y)$ is quasi-monotone non-decreasing in x and non-decreasing in y iff it is mixed quasi-monotone non-decreasing in x, non-decreasing in y.*

Proof Omitted because obvious.

Example 4.2 If we consider

$$g\colon \mathbb{R}\times\mathbb{R}^n\times\mathcal{C}\to\mathbb{R}^n$$
$$(t,x,x_t)\mapsto\begin{pmatrix}a_{11}(x_1) & a_{12}(t)\\ a_{21}(t) & a_{22}(x_2)\end{pmatrix}x+\begin{pmatrix}b_{11}(t) & b_{12}(t)\\ b_{21}(t) & b_{22}(t)\end{pmatrix}x(t-\tau), \tag{21}$$

with $\forall t\in\mathbb{R}$, $a_{21}(t)\ge 0$, $a_{12}(t)\ge 0$, $\forall(i,j)\in\{1,2\}^2\colon b_{ij}(t)\ge 0$, then g is quasi-monotone non-decreasing in x and non-decreasing in x_t.

Lemma 4.2 *If g is continuously differentiable with respect to x and y, and*

$$\forall t\in\mathbb{R},\quad \forall x\in\mathbb{R}^n,\quad \forall y\in\mathcal{C}$$
$$\forall i\ne j\colon \frac{\partial g_i}{\partial x_j}\ge 0, \tag{22}$$
$$\forall(i,j)\colon \frac{\partial g_i}{\partial y_j}\ge 0, \tag{23}$$

then $g(t,x,y)$ is mixed quasi-monotone non-decreasing in x, non-decreasing in y.

Proof Using Lemma 4.1, let us first prove that g is quasi-monotone non-decreasing in x.

Let $y\in\mathcal{C}$, and consider any $(x,x')\in\mathbb{R}^n\times\mathbb{R}^n$ such that $\exists i\in\{1,\dots,n\}$: $(x_i=x_i')\wedge(x\le x')$. Define Θ as:

$$\Theta\colon [0,1]\to\mathbb{R}^n$$
$$\theta\mapsto x+\theta(x'-x).$$

Naturally $g_i(t,\Theta(\theta),y)$ is continuously differentiable in its second variable, so

$$g_i(t,\Theta(1),y)-g_i(t,\Theta(0),y) = \int_0^1\sum_{j=1}^n\left(\frac{\partial g_i(t,x,y)}{\partial x_j}\right)\bigg|_{x=x+\theta(x'-x)}(x_i'-x_i)\,d\theta\le 0. \tag{24}$$

The fact that g is non-decreasing in y may be proved in the same way.

Remark 4.2 In (23), y_j is a function and the map g_i is a functional.

It is now possible to state the main result of this subsection: a comparison principle for functional differential equations

Theorem 4.1 *Assume that:*

H1) $\forall t\in\mathbb{R}$, $\forall x\in\mathbb{R}^n$, $\forall y\in\mathcal{C}\colon f(t,x,y)\le g(t,x,y)$,

H2) *$g(t,x,y)$ is mixed quasi-monotone non-decreasing in x, non-decreasing in y*

H3) *$g(t,x,y)$ is sufficiently smooth for (13) to possess, for every $z_{t_0}\in\Omega\subset\mathcal{C}$ and for every $t_0\in\mathbb{R}$, a unique solution $z(t)$ for all $t\ge t_0$.*

Then:

C1) *For any $z_{t_0}\in\Omega$, the inequality*

$$x(t)\le z(t), \tag{25}$$

holds for every $t\ge t_0$ whenever it is satisfied for $t\in[t_0-\tau,t_0]$. In other words, (13) is a comparison system of (12) over Ω.

C2) *Moreover, if $\forall t\ge t_0\colon 0\le g(t,0,\varphi_0)$ and $z_{t_0}\ge 0$, then $0\le z(t)$.*

Proof Let z^k be a solution of the system

$$\dot{z}^k(t) = g(t, z^k(t), z_t^k) + \delta_k, \tag{26}$$

with the initial condition

$$z_{t_0}^k = z_{t_0} + \varphi_{\delta_k}, \tag{27}$$

$$z_{t_0} \in \Omega \subset \mathcal{C}, \tag{28}$$

where δ_k is the n-vector each component of which is equal to $\left(\frac{1}{2}\right)^k$, and φ_{δ_k} is the constant function defined on $[-\tau, 0]$ of value δ_k.

By H3) $z^k(t)$ is defined and continuous with respect to time for all $t \geq t_0$, and naturally, the sequence $z^k(t)$ converges uniformly towards the solution $z(t)$ of (13) with the initial condition $z_{t_0} \in \Omega \subset \mathcal{C}$.

C1) $x(t) < z^k(t)$ holds for every $t \geq t_0$ if $z_{t_0} \geq x_{t_0}$ (both in $\Omega \subset \mathcal{C}$): otherwise, there would be a time $t_1 > t_0$ and an index $i_0 \in \{1, \ldots, n\}$ such that

$$x_{i_0}(t_1) = z_{i_0}^k(t_1). \tag{29}$$

Then, let $t_2 = \inf\{t \geq t_0 \colon \exists i \in \{1, ..., k\} \colon x_i(t) = z_i^k(t)\}$. By continuity and (27), we have $t_2 > t_0$. By denoting $\varepsilon(t) = z^k(t) - x(t)$, we then obtain from (26) and H1):

$$\dot{\varepsilon}_i(t_2) \geq g(t_2, z^k(t_2), z_{t_2}^k) - g(t_2, x(t_2), x_{t_2}) + \left(\frac{1}{2}\right)^k, \tag{30}$$

But, at time $t = t_2$, we have:

$$\begin{aligned} x_{i_0}(t_2) &= z_{i_0}^k(t_2), \\ z_i^k(t_2) &\geq x_i(t_2), \quad \forall i \neq i_0, \\ z_{t_2}^k &\geq x_{t_2}. \end{aligned} \tag{31}$$

According to H2), we have $g(t_2, z^k(t_2), z_{t_2}^k) - g(t_2, x(t_2), x_{t_2}) \geq 0$:

$$\dot{\varepsilon}_i(t_2) \geq \left(\frac{1}{2}\right)^k > 0. \tag{32}$$

It follows that function $\varepsilon_i(t)$ is increasing on a neighborhood of t_2. But, by definition of t_2, we have:

$$\dot{\varepsilon}_{i_0}(t_2^-) = \lim_{\theta \to 0-} \frac{\varepsilon_{i_0}(t_2 + \theta) - \varepsilon_{i_0}(t_2)}{\theta} \leq 0, \tag{33}$$

which leads to a contradiction and completes the proof.

C2) In the same way: $\dot{z}_i^k(t_2) = g(t_2, z^k(t_2), z_{t_2}^k)) + \left(\frac{1}{2}\right)^k$ using H2) and $\forall t \geq t_0 \colon 0 \leq g(t, 0, \varphi_0)$ leads to $\dot{z}_i^k(t_2) \geq \left(\frac{1}{2}\right)^k > 0$, which contradicts

$$\dot{z}_{i_0}^k(t_2^-) = \lim_{\theta \to 0-} \frac{z_{i_0}^k(t_2 + \theta) - z_{i_0}^k(t_2)}{\theta} \leq 0.$$

Remark 4.3 One can refine the definitions given above by considering local comparison system and thus obtain a local version of this theorem (see [37, 39]).

Remark 4.4 Note that for ordinary differential systems, one can see that the given conditions are reduced to the well-known Wazewski conditions (see [50]).

Corollary 4.1 *Assume that:*

H1) $\forall t \in \mathbb{R}$, $\forall x \in \mathbb{R}_+^n$, $\forall y \in \mathcal{C}_+ = \{y \in \mathcal{C} \colon y(s) \geq 0, s \in [-\tau, 0]\} \colon 0 \leq g(t, 0, 0)$,

H2) $g(t, x, y)$ *is mixed quasi-monotone non-decreasing in* x, *non-decreasing in* y,

H3) $g(t, x, y)$ *is sufficiently smooth for (13) to possess, for every* $z_{t_0} \in \Omega \subset \mathcal{C}$ *and for every* $t_0 \in \mathbb{R}$, *a unique solution* $z(t)$ *for all* $t \geq t_0$.

Then, for any $z_{t_0} \in \mathcal{C}_+$, *the inequality*

$$0 \leq z(t), \tag{34}$$

holds for every $t \geq t_0$.

4.2 Vector Lyapunov functions

The concept of vector Lyapunov function (VLF) ([1, 31 − 33, 37]) and vector norms (VN) ([4, 6, 8]) have been widely developed in the literature. Here, we shall consider only the following definitions.

Definition 4.4 V is said to be a *vector Lyapunov function* if:

$$\begin{aligned} &V \colon \mathbb{R}^n \to \mathbb{R}^k \\ &x \mapsto V(x) = [v_1(x), \ldots, v_k(x)]^T, \end{aligned} \tag{35}$$

where $v_i(x)$ are continuous, semi-positive definite functions and $[V(x) = 0 \Leftrightarrow x = 0]$.

If there is a scalar $c > 0$ such that, for all vector $x \in \mathbb{R}^n$, $\|V(x)\|_k \geq c\|x\|_n$ where $\|\cdot\|_k$ (respectively $\|\cdot\|_n$) denotes any given norm on $\mathbb{R}^k$ (resp. $\mathbb{R}^n$), then V is said to be *radially unbounded*.

Example 4.3 The map

$$V : \mathbb{R}^3 \to \mathbb{R}^2_+$$
$$x \mapsto V(x) = [x_1^2 + x_2^2,\ (x_2 - x_3)^2 + x_3^2]^T \tag{36}$$

is a VLF whereas $[x_1^2 + 2x_1x_2 + x_2^2,\ (x_2 - x_3)^2]^T$ is not a VLF.

In the following definitions, it is assumed that

$$\mathbb{R}^n = \bigoplus_{i=1}^{k} E_i, \tag{37}$$

where E_i are vector subspaces of $\mathbb{R}^n$ with $\dim(E_i) = n_i$.

Definition 4.5 P is said to be a *regular vector norm* if:

$$P : \mathbb{R}^n \to \mathbb{R}^k_+$$
$$x \mapsto P(x) = [p_1(x^{[1]}), \ldots, p_k(x^{[k]})]^T, \tag{38}$$

where $p_i(x^{[i]})$ are vector norms (in the classical sense) on E_i and $x^{[i]} = \mathrm{Pr}_i(x)$ is the projection of x onto E_i. If the sum (37) is not direct, then P is said to be a *vector norm.*

Remark 4.5 If P is a vector norm, then it is also a vector Lyapunov function. Moreover, vector norms satisfy properties similar to those of classical scalar norms.

Example 4.4 Consider

$$P : \mathbb{R}^3 \to \mathbb{R}^2_+$$
$$x \mapsto V(x) = [|x_1| + |x_2| + |x_3|,\ |x_2|]^T, \tag{39}$$

is a VN whereas $[|x_1| + |x_2|,\ |x_3|]^T$ is a regular VN.

Lemma 4.3 *$P(x)$ is radially unbounded and for $c > 0$, $\bar{\mathcal{X}}_c = \{x \in \mathbb{R}^n : P(x) \le c\}$ are compact connected sets.*

4.3 Construction of comparison systems

One of the main advantages in using regular vector norms as particular vector Lyapunov functions is to allow for the computation of comparison systems in an easy and systematic way for a wide class of systems (7).

In this section, we assume that the mapping f appearing in (7) has the form:

$$f(t, x(t), x_t, d) = A(t, x(t), x_t, d)\, x(t) + B(t, x(t), x_t, d)\, x_t + c(t, x(t), x_t, d), \tag{40}$$

where for any given $(t, x(t), x_t, d)$ (which will be shortened to $(.)$ in the sequel) $A(.)$ is a $(n \times n)$-matrix with real entries, the map

$$B(.) : \mathcal{C} \to \mathbb{R}^n$$
$$\varphi \mapsto B(.)\varphi$$

is linear in φ and bounded, and $c(.)$ is a n-vector.

4.3.1 General expressions. Let us calculate the right-hand Dini derivative c
$p_i(x^{[i]}(t))$ with respect to time for $1 \le i \le k$:

$$\begin{aligned} D^+ p_i(x^{[i]}(t)) &= (\text{grad } p_i(x^{[i]}(t)))^T \dot{x}^{[i]}(t) = (\text{grad } p_i(x^{[i]}(t)))^T \mathrm{Pr}_i\, \dot{x}(t) \\ &= (\text{grad } p_i(x^{[i]}(t)))^T \mathrm{Pr}_i [A(.)\, x(t) + B(.)\, x_t + c(.)] \\ &= \sum_{j=1}^{k} (\text{grad } p_i(x^{[i]}(t)))^T \mathrm{Pr}_i\, A(.)\, \mathrm{Pr}_j\, x^{[j]}(t) \\ &\quad + \sum_{j=1}^{k} (\text{grad } p_i(x^{[i]}(t)))^T \mathrm{Pr}_i\, B(.)\, \mathrm{Pr}_j \circ x_t \\ &\quad + (\text{grad } p_i(x^{[i]}(t)))^T \mathrm{Pr}_i\, c(.), \end{aligned} \tag{41}$$

Let us consider each term separately.

First term: If $p_j(x^{[j]}(t)) \neq 0$, then

$$\begin{aligned} &(\text{grad } p_i(x^{[i]}(t)))^T \mathrm{Pr}_i\, A(.)\, \mathrm{Pr}_j\, x^{[j]}(t) \\ &= \frac{(\text{grad } p_i(x^{[i]}(t)))^T \mathrm{Pr}_i\, A(.)\, \mathrm{Pr}_j\, x^{[j]}(t)}{p_j(x_j(t))}\, p_j(x^{[j]}(t)) \le m_{ij}(.)\, p_j(x^{[j]}(t)), \end{aligned} \tag{42}$$

where

$$m_{ij}(.) = \sup_{u \in \mathbb{R}^n} \left\{ \frac{\text{grad } p_i(u^{[i]})^T \mathrm{Pr}_i\, A(.)\, \mathrm{Pr}_j\, u^{[j]}}{p_j(u^{[j]})} \right\}. \tag{43}$$

Note that the previous inequality still holds if $p_j(x^{[j]}(t)) = 0$.

Second term: The mapping $\varphi \mapsto (\text{grad } p_i(x^{[i]}(t)))^T \mathrm{Pr}_i\, B(.)\, \mathrm{Pr}_j \circ \varphi$ is a bounde
functional. So, using Riesz's representation of linear functional by Stieljes integr
[44], we have

$$(\text{grad } p_i(x_i(t)))^T \mathrm{Pr}_i\, B(.)\, \mathrm{Pr}_j \circ \varphi = \sum_{l=1}^{n_j} \int_{-\tau}^{0} \mathrm{d}k_{ijl}(.)(s)\, \varphi_{jl}(s), \tag{44}$$

where $k_{ijl}(.)$ are real-valued functions of bounded variation defined on $[-\tau, 0]$, and where $\varphi_j = [\varphi_{j1}, \dots, \varphi_{jn_j}]^T$ denotes $\mathrm{Pr}_j \circ \varphi$. Each function $k_{ijl}(.)$ may be written as the difference between two non-decreasing functions [44]:

$$k_{ijl}(.) = \alpha_{ijl}(.) - \beta_{ijl}(.).$$

By defining $k'_{ijl}(.) = \alpha_{ijl}(.) + \beta_{ijl}(.)$, we have the inequalities

$$\begin{aligned} |(\text{grad } p_i(x^{[i]}(t)))^T \,\mathrm{Pr}_i\, B(.)\, \mathrm{Pr}_j \circ \varphi| &\le \sum_{l=1}^{n_j} \int_{-\tau}^{0} \mathrm{d}k'_{ijl}(.)(s)\, |\varphi_{jl}(s)| \\ &\le \int_{-\tau}^{0} \sum_{l=1}^{n_j} \mathrm{d}k'_{ijl}(.)(s) \max_{1 \le l \le n_j} |\varphi_{jl}(s)| \end{aligned}$$

and then, since norms on E_j are equivalent, for each j, there is a real number $\sigma_j > 0$ such that

$$|(\text{grad } p_i(x^{[i]}(t)))^T \,\mathrm{Pr}_i\, B(.)\, \mathrm{Pr}_j \circ \varphi| \le \int_{-\tau}^{0} \sum_{l=1}^{n_j} \sigma_j \,\mathrm{d}k'_{ijl}(.)(s)\, p_j(\varphi_j(s)).$$

Denoting $n_{ij}(.)$ the functional defined on $C([-\tau, 0], \mathbb{R})$ by

$$n_{ij}(.) \colon \psi \mapsto n_{ij}(.)\psi = \int_{-\tau}^{0} \sum_{l=1}^{n_j} \sigma_j \,\mathrm{d}k'_{ijl}(.)(s)\, \psi(s), \tag{45}$$

we have

$$|(\text{grad } p_i(x^{[i]}(t)))^T \,\mathrm{Pr}_i\, B(.)\, \mathrm{Pr}_j \circ \varphi| \le n_{ij}(.)\, p_j \circ \varphi_j.$$

Third term: Let us denote

$$q_i(.) = |(\text{grad } p_i(x^{[i]}(t)))^T \,\mathrm{Pr}_i\, c(.)|, \tag{46}$$

Then, summarizing the previous steps, the vector $D^+P(x(t))$ satisfies the following inequality:

$$D^+P(x(t)) \le M(.)\, P(x(t)) + N(.)\, P \circ x_t + q(.),$$

where $M(.) = \{m_{ij}(.)\}$ is a $(k \times k)$ matrix, $N(.)$ is the mapping $\mathcal{C} \to \mathbb{R}^k$ given by $N(.)\varphi = \left[\sum_{j=1}^{k} n_{1j}(.) p_j \circ \varphi_j, \dots, \sum_{j=1}^{k} n_{kj}(.) p_j \circ \varphi_j\right]^T$, and $q(.)$ denotes the k-vector $[q_1(.), \dots, q_k(.)]^T$.

It only remains to be proved that the system

$$\dot z(t) = M(.)\, z(t) + N(.)\, z_t + q(.)$$

is a comparison system for the initial one: according to Theorem 4.1, it is enough to prove that the function $g(t, x, y) = M(.)\, x + N(.)\, y + q(.)$ is mixed quasi-monotone non-decreasing in x, non-decreasing in y. This is obvious if we observe that the off-diagonal elements of $M(.)$ are non-negative (in expression (43), the subvectors $u^{[i]}$ and $u^{[j]}$ can be chosen freely).

4.3.2 Particular cases for the construction of comparison systems. Simpler expressions may be found under stronger assumptions about the choice of the norms and the initial system.

Let us now consider a system of the form

$$\begin{aligned} \dot x(t) = {} & A(t, x(t), x_t, d)\, x(t) + \sum_{k=1}^{m} B^k(t, x(t), x_t, d)\, x(t - \tau_k(t)) \\ & + \int_{-\tau}^{0} K(s, t, x(t), x_t, d)\, x(t+s)\, \mathrm{d}s + c(t, x(t), x_t, d), \end{aligned} \tag{47}$$

where $A(.)$ and $B^k(.)$ (for $k = 1, \dots, m$) and $K(s, t, x(t), x_t, d)$ are $n \times n$ matrices, and $\tau_k(t)$ are positive, bounded and piecewise-continuous functions such that

$$0 \le \tau_k(t) \le \tau, \quad \forall\, t \ge t_0, \quad \forall\, k = 1, \dots, m.$$

To simplify, we consider here that the subspaces E_i of the partition (37) are given by

$$E_i = \mathrm{span}\,\{e_j : j \in \mathcal{I}_i\}, \quad i = 1, \dots, k,$$

where $\{e_1, e_2, \dots, e_n\}$ denotes the canonical basis of $\mathbb{R}^n$ and where the $\mathcal{I}_i$'s are disjoint subsets of $\mathbb{N}_n = \{1, 2, \dots, n\}$ such that

$$\mathbb{N}_n = \bigcup_i \mathcal{I}_i. \tag{48}$$

The norm p_i on the subspace E_i will be defined by $p_i(x^{[i]}) = \|x^{(i)}\|_i$, where $x^{(i)}$ is the n_i-vector of components x_j for $j \in \mathcal{I}_i$ and $\|\cdot\|_i$ is a given norm on $\mathbb{R}^{n_i}$.

According to the previous partition (48), we associate with a given matrix $A = \{a_{ij}\} \in \mathbb{R}^{n \times n}$, the k^2 matrices A_{ij} defined by

$$A_{ij} = \{a_{lm}\}_{\substack{l \in \mathcal{I}_i \\ m \in \mathcal{I}_j}} \qquad \text{for } (i, j) \in (\mathbb{N}_k)^2.$$

Then, a comparison system for system (47) is

$$\dot z(t) = \Gamma(A(t,x(t),x_t,d))\, z(t) + \sum_{k=1}^{m} \Lambda(B^k(t,x(t),x_t,d))\, z(t-\tau_k(t)) + \int_{-\tau}^{0} \Lambda(K(s,t,x(t),x_t,d))\, z(t+s)\, \mathrm{d}\, s + P(c(t,x(t),x_t,d)), \tag{49}$$

with the following notations

- $\Gamma(A)$ is the $(k\times k)$-matrix associated with $A=\{a_{ij}\}\in\mathbb{R}^{n\times n}$ by

$$\Gamma(A)_{lm} = \begin{cases} \mu_l(A_{ll}) & \text{for } l=m \\ \|A_{lm}\|_{lm} = \max\limits_{0\neq y\in R^{n_m}} \frac{\|A_{lm}y\|_l}{\|y\|_m} & \text{for } l\neq m, \end{cases} \tag{50}$$

where $\mu_l(A_{ll})$ is the measure of the matrix A_{ll} associated with the norm $\|\cdot\|_l$ and is defined by the expression

$$\mu(A_{ll}) = \lim_{\varepsilon\to 0^+} \frac{\|I_{n_l}+\varepsilon\, A_{ll}\|_l - 1}{\varepsilon} \tag{51}$$

(we refer to [15] for more details about properties of matrix measures).

- $\Lambda(A)$ is the $(k\times k)$-matrix associated with $A=\{a_{ij}\}\in\mathbb{R}^{n\times n}$ by

$$\Lambda(A)_{lm} = \|A_{lm}\|_{lm} \quad \text{for } l,m=1,\dots,k. \tag{52}$$

This result may be found in another form in [46] or [47] and generalizes the formulas given in [13] (see also [12] and [10]).

We only outline the proof of this result. By continuity, we have

$$\begin{aligned} D^+p_i(x^{[i]}(t)) &= \lim_{\varepsilon\to 0^+} \frac{p_i(x^{[i]}(t+\varepsilon)) - p_i(x^{[i]}(t))}{\varepsilon} \\ &= \lim_{\varepsilon\to 0^+} \frac{\|x^{(i)}(t)+\varepsilon\dot x^{(i)}(t)\|_i - \|x^{(i)}(t)\|_i}{\varepsilon}. \end{aligned} \tag{53}$$

Then, from (47) and the properties of the norm $\|\cdot\|_i$, this gives

$$\begin{aligned} D^+p_i(x^{[i]}(t)) \le{}& \lim_{\varepsilon\to 0^+}\left(\frac{\|I_{n_i}+\varepsilon A_{ii}\|-1}{\varepsilon}\right)\|x^{(i)}(t)\|_i \\ &+ \sum_{j\neq i} \|A_{ij}(.)\|_{ij}\|x^{(j)}(t)\|_j \\ &+ \sum_k\sum_j \|B^k_{ij}(.)\|_{ij}\|x^{(j)}(t-\tau_k(t))\|_j \\ &+ \sum_j \int_{-\tau}^{0} \|K_{ij}(.)\|_{ij}\|x^{(j)}(t+s)\|_j\ \mathrm{d}s + \|c^{(i)}(.)\|_i. \end{aligned} \tag{54}$$

The result follows from definition (51).

There are simple and explicit expressions of the terms appearing in (50) and (52) in the case of usual norms. Let $A=\{a_{ij}\}$ denote a $(m\times n)$-matrix, $B=\{b_{ij}\}$ a $(n\times n)$-matrix, x a n-vector and y a m-vector.

With $p_1(x)=\|x\|_1 \triangleq \sum_{i=1}^{n}|x_i|$, $p_2(x)=\|x\|_\infty \triangleq \max_{1\le i\le n}|x_i|$, we have:

$$\begin{aligned} &\|A\|_{11} = \max_{j\in\{1,\dots,n\}} \sum_{i=1}^{m}|a_{ij}|, \qquad \|A\|_{12} \le \sum_{i=1}^{m}\sum_{j=1}^{n}|a_{ij}|, \\ &\|A\|_{22} = \max_{i\in\{1,\dots,m\}} \sum_{j=1}^{n}|a_{ij}|, \qquad \|A\|_{21} = \max_{\substack{1\le i\le m\\ 1\le j\le n}} |a_{ij}|, \\ &\mu_1(B) = \max_j \Big[\Re(b_{jj}) + \sum_{i\neq j}|b_{ij}|\Big], \quad \mu_2(B) = \max_i \Big[\Re(b_{ii}) + \sum_{j\neq i}|b_{ij}|\Big]. \end{aligned} \tag{55}$$

It is possible to obtain other and simpler comparison systems at the expense of stronger estimations. Indeed, for any matrix $\tilde M(.)$, any mapping $\tilde N(.)\colon \mathcal{C}(\mathbb{R}^k)\to\mathbb{R}^k$, and any k-vector $\tilde q(.)$ such that:

$$\begin{aligned} &\tilde M(.) \ge M(.), \\ &\tilde N(.)\varphi \ge N(.)\varphi, \quad \forall\varphi\in\mathcal{C}(\mathbb{R}^k), \\ &\tilde q(.) \ge q(.), \end{aligned} \tag{56}$$

the system

$$\dot z(t) = \tilde M(.)\, z(t) + \tilde N(.)\, z_t + \tilde q(.)$$

is also a comparison system of (47). For instance, when t does not appear explicitly, one can find a neighbourhood $\mathcal{N}$ of the origin $(0,0)\in\mathbb{R}^k\times\mathcal{C}(\mathbb{R}^k)$ and a map g mixed quasi-monotone non-decreasing in $P(x(t))$, non-decreasing in $P\circ x_t$, such that

$$g(P(x(t)), P\circ x_t) \ge M(x(t),x_t)P(x(t)) + N(x(t),x_t)P\circ x_t + q(x(t),x_t) \tag{57}$$

holds on $\mathcal{N}$, thus leading to a local CS $\dot z(t) = g(z(t),z_t)$.

Note that when M (constant) is Hurwitz (that is, all eigenvalues are in the open left-half plane), the opposite of M then belongs to the class of M-matrices (see [17]).

5 Qualitative Properties of Sets

In this part, $\mathcal{A}$ denotes a non-empty compact set which can be restricted to a point.

5.1 Stability, attractivity

5.1.1 Definitions. In this subsection, we formally define or recall different qualitative properties of sets such as stability, attractivity and so on. Moreover, as the dynamical description of the system is time-dependent, the same should be true for the definition (as mentioned and illustrated in [19]). Lastly, we will stress the importance of having information about initial conditions leading to a qualitative property for a solution: this is the notion of domain relative to a property.

Definition 5.1 The *set* $\mathcal{A} \subset \mathcal{C}$ is said to be *stable w.r.t.* $\{t_0\}$ for system (7) if

$$\mathcal{P}_s(t_0):\begin{cases}\forall\varepsilon>0,\exists\delta(t_0,\varepsilon)>0:\\ \varphi\in B(\mathcal{A},\delta(t_0,\varepsilon))\Rightarrow x_t(t_0,\varphi)\in B(\mathcal{A},\varepsilon),\ \forall t\geq t_0,\end{cases}\tag{58}$$

holds, *stable w.r.t.* $\mathcal{T}$ if $\mathcal{P}_s(\mathcal{T})$: $[\forall t_0 \in \mathcal{T}, \mathcal{P}_s(t_0)]$ holds, *stable* if $\mathcal{P}_s(\mathbb{R})$: $[\forall t_0 \in \mathbb{R}, \mathcal{P}_s(t_0)]$ holds. These properties are said to be *uniform* if in $\mathcal{P}_s(t_0)$, $\delta(t_0,\varepsilon) = \delta(\varepsilon)$ does not depend on t_0, and the corresponding properties are denoted by $\mathcal{P}_{us}(.)$, where (.) is respectively $t_0, \mathcal{T}, \mathbb{R}$.

Definition 5.2 The *set* $\mathcal{A} \subset \mathcal{C}$, is said to be *exponentially stable w.r.t.* $\{t_0\}$ for system (7) if

$$\mathcal{P}_{es}(t_0):\begin{cases}\exists\delta(t_0)>0,\ \exists\alpha(t_0)>0,\ \exists\beta(t_0)>0:\\ \varphi\in B(\mathcal{A},\delta(t_0))\Rightarrow x_t(t_0,\varphi)\in B(\mathcal{A},\beta\exp(-\alpha(t-t_0))\rho(\varphi,\mathcal{A})),\\ \qquad\forall t\geq t_0\end{cases}\tag{59}$$

holds, *exponentially stable w.r.t.* $\mathcal{T}$ if $\mathcal{P}_{es}(\mathcal{T})$: $[\forall t_0 \in \mathcal{T}, \mathcal{P}_{es}(t_0)]$ holds, *exponentially stable* if $\mathcal{P}_{es}(\mathbb{R})$: $[\forall t_0 \in \mathbb{R}, \mathcal{P}_s(t_0)]$ holds. These properties are said to be *uniform* if in $\mathcal{P}_s(t_0)$, $\delta(t_0) = \delta$, $\alpha(t_0) = \alpha$, and $\beta(t_0) = \beta$, i.e., they do not depend on t_0, the corresponding properties are denoted by $\mathcal{P}_{ue}(.)$, where (.) is respectively t_0, $\mathcal{T}$, $\mathbb{R}$.

Definition 5.3 The *set* $\mathcal{A} \subset \mathcal{C}$, is said to be *attractive w.r.t.* $\{t_0\}$ for system (7) if

$$\mathcal{P}_a(t_0):\begin{cases}\forall\varepsilon>0,\ \exists\delta(t_0,\varepsilon)>0,\ \exists T(t_0,\varepsilon)\geq t_0:\\ \varphi\in B(\mathcal{A},\delta(t_0,\varepsilon))\Rightarrow x_t(t_0,\varphi)\in B(\mathcal{A},\varepsilon),\ \forall t\geq T(t_0,\varepsilon)\end{cases}\tag{60}$$

holds, *attractive w.r.t.* $\mathcal{T}$ if $\mathcal{P}_a(\mathcal{T})$: $[\forall t_0 \in \mathcal{T}, \mathcal{P}_a(t_0)]$ holds, *attractive* if $\mathcal{P}_a(\mathbb{R})$: $[\forall t_0 \in \mathbb{R}, \mathcal{P}_a(t_0)]$ holds. These properties are said to be *uniform* if in $\mathcal{P}_a(t_0)$, $\delta(t_0,\varepsilon) = \delta(\varepsilon)$ and $T(t_0,\varepsilon) = T(\varepsilon)$ does not depend on t_0, the corresponding properties are denoted by $\mathcal{P}_{ua}(.)$, where (.) is respectively t_0, $\mathcal{T}$, $\mathbb{R}$.

Definition 5.4 The set $\mathcal{A} \subset \mathcal{C}$ is said to be *asymptotically stable w.r.t.* $\{t_0\}$ (respectively *asymptotically stable w.r.t.* $\mathcal{T}$, *asymptotically stable*) for system (7) if $\mathcal{A}$ is stable w.r.t. $\{t_0\}$ and attractive w.r.t. $\{t_0\}$ (respectively stable w.r.t. $\mathcal{T}$ and attractive w.r.t. $\mathcal{T}$, stable and attractive). These properties are said to be *uniform* if attractivity and stability are uniform.

From a practical point of view, it is crucial in engineering sciences to have concrete knowledge of the initial sets implied in these definitions (in other words, to have information about the number δ appearing in the previous definitions). This induces the concept of domain relative to some property (see [19, 22]). The proposed definitions are inspired by those given for the equilibrium point of an ordinary differential equation ([19, 22]) and extended to sets in the case of ordinary differential equations [37]: here, we define them for the general class of functional differential equations.

Definition 5.5 The *domain of stability* $\mathcal{D}_s(t_0,\mathcal{A}) \subset \mathcal{C}$ of a non-empty compact set $\mathcal{A} \subset \mathcal{C}$ *w.r.t.* $\{t_0\}$ for system (7) is defined by:

(1) $\forall\varepsilon > 0, x_t(t_0,\varphi)) \in B(\mathcal{A},\varepsilon)$, $\forall t \geq t_0$, iff $\varphi_{x_0} \in \mathcal{D}_s(t_0,\mathcal{A})$,
(2) $\mathcal{D}_s(t_0,\mathcal{A})$ is a neighbourhood of $\mathcal{A}$.

The *domain of stability* $\mathcal{D}_s(\mathcal{T},\mathcal{A}) \subset \mathcal{C}$ of a non-empty compact set $\mathcal{A} \subset \mathcal{C}$ *w.r.t.* $\mathcal{T}$ is defined by:

(1) $\forall t_0 \in \mathcal{T}, \mathcal{D}_s(t_0,\mathcal{A})$ exists,
(2) $\mathcal{D}_s(\mathcal{T},\mathcal{A}) = \cap_{t_0\in\mathcal{T}}\mathcal{D}_s(t_0,\mathcal{A})$.

The *domain of stability* of a non-empty compact set $\mathcal{A} \subset \mathcal{C}$ is $\mathcal{D}_s(\mathbb{R},\mathcal{A})$.

The other *domains of uniform stability* $\mathcal{D}_{us}(.,\mathcal{A})$ are similarly defined, where (.) stands for t_0, $\mathcal{T}$, or $\mathbb{R}$.

The other domains corresponding to all the previously mentioned qualitative properties can be defined in a similar way and are omitted for sake of brevity. Note that $\mathcal{D}_{as}(.,\mathcal{A}) = \mathcal{D}_s(.,\mathcal{A}) \cap \mathcal{D}_a(.,\mathcal{A})$, where (.) is respectively $t_0, \mathcal{T}, \mathbb{R}$.

5.1.2 Results. This part concerns the qualitative properties defined in the previous part that can be deduced from the construction of a comparison system, as described in Section 4. Thus, we consider that the techniques described above lead to the following functional differential inequality

$$D^+P(x) \leq g(P(x(t)), P(x_t)),\tag{61}$$

with which we associate

$$\dot{z}(t) = g(z(t), z_t).\tag{62}$$

Note that this is at least possible locally (see Section 4.3).

Let us recall that in that case if $g(0,0) \geq 0$, then Corollary 4.1 can be applied and the estimating relation $0 \leq P(x(t)) \leq z(t)$ holds for $t > t_0$ as soon as it is satisfied for $t \in [t_0 - \tau, t_0]$. Using this fact, we obtain:

Theorem 5.1 *Let $\mathcal{P}$ be one of the properties defined in the previous subsection. Assume that system (62) has a positive equilibrium point z_e with a non-empty domain $\mathcal{D}_{\mathcal{P}}(\varphi_{z_e})$, then $\mathcal{A} = \{\varphi \in \mathcal{C}\colon P(\varphi(s)) \leq z_e, \forall\, s \in [-\tau, 0]\}$ has property $\mathcal{P}$ with the associated domain estimated by $\mathcal{D}_{\mathcal{P}}(\mathcal{A}) = \{\varphi \in \mathcal{C}\colon P(\varphi(s)) \in \mathcal{D}_{\mathcal{P}}(z_e), \forall\, s \in [-\tau, 0]\}$.*

Remark 5.1 For VLF, one can obtain a result similar to Theorem 5.1, as soon as g in (61) is mixed quasi-monotone non-decreasing in $V(x(t))$, non-decreasing in $V(x_t)$ and V is radially unbounded.

Example 5.1 Consider the system defined by

$$\begin{cases} \dot{x}_1 = (1 - x_1^2 - x_2^2)x_1 + d_{12}(t)x_2 + \int\limits_{t-\tau}^{t} d_{11}(s)x_1(s)\,\mathrm{d}s, \\ \dot{x}_2 = (1 - x_1^2 - x_2^2)x_2 + d_{21}(t)x_1 + \int\limits_{t-\tau}^{t} d_{22}(s)x_2(s)\,\mathrm{d}s, \end{cases} \tag{63}$$

where $d_{ij}(t)$ are continuous functions, bounded in magnitude by 1, and let us consider the vector norm

$$P(x) = \begin{pmatrix} |x_1| \\ |x_2| \end{pmatrix}. \tag{64}$$

We then have

$$D^+P(x) \leq \begin{pmatrix} (1 - P_1^2(x) - P_2^2(x)) & 1 \\ 1 & (1 - P_1^2(x) - P_2^2(x)) \end{pmatrix} P(x) + \int\limits_{t-\tau}^{t} P(x(s))\,\mathrm{d}s. \tag{65}$$

We can consider

$$\dot{z}(t) = g(z(t), z_t) = \begin{pmatrix} 1 - z_1^2(t) & 1 \\ 1 & 1 - z_2^2(t) \end{pmatrix} z(t) + \int\limits_{t-\tau}^{t} z(s)\,\mathrm{d}s, \tag{66}$$

to be a comparison system (g satisfies the hypotheses of Corollary 4.1). System (66) has an equilibrium point defined by

$$z_e = \sqrt{2+\tau}\,(1,1)^T. \tag{67}$$

A change of variable ($y + z_e = z$) leads to the system

$$\dot{y}(t) = \begin{pmatrix} -(5+3\tau) - 3\sqrt{2+\tau}y_1 - y_1^2(t) & 1 \\ 1 & -(5+3\tau) - 3\sqrt{2+\tau}y_1 - y_1^2(t) \end{pmatrix} y(t) + \int\limits_{t-\tau}^{t} y(s)\,\mathrm{d}s. \tag{68}$$

Let us consider the function $v(y) = \max(|y_1|, |y_2|)$ and, following Razumikhin's method [41], assume that:

$$v(y(t+s)) < (1+\varepsilon)\,v(y(t)), \quad \forall\, s \in [-\tau, 0]$$

with $\varepsilon > 0$.

There is an index $i \in \{1,2\}$, such that $v(y(t)) = |y_i(t)|$ and $\dot{v}(y(t)) = \operatorname{sign}(y_i(t))\,\dot{y}_i(t)$. It follows that

$$\dot{v}(y(t)) \leq (-(4+2\tau) + \varepsilon - 3\sqrt{2+\tau}y_i - y_i^2(t))\,v(y(t)).$$

For any $y(t)$ such that $y_1 > -\sqrt{(2+\tau)}$ and $y_2 > -\sqrt{(2+\tau)}$, it is possible to choose ε such that $\dot{v}(y(t)) < 0$.

This proves that z_e is asymptotically stable with $\mathcal{D}_{as}(\varphi_{z_e}) = \mathcal{C}_+ \setminus \{0\}$. Then, according to Theorem 5.1, the set $\mathcal{A} = \{\varphi \in \mathcal{C}\colon P(\varphi(s)) \leq z_e, \forall\, s \in [-\tau, 0]\}$ is globally asymptotically stable.

Note that in the particular, ordinary, case $\tau = 0$ we find again the results proved in [37, 39], stating that $\mathcal{A} = \{x \in \mathbb{R}^n\colon P(x) \leq \sqrt{2}(1,1)^T\}$ is globally asymptotically stable for the corresponding ordinary differential system:

$$\begin{cases} \dot{x}_1 = (1 - x_1^2 - x_2^2)x_1 + d_{12}(t)x_2 \\ \dot{x}_2 = (1 - x_1^2 - x_2^2)x_2 + d_{21}(t)x_1 \end{cases}. \tag{69}$$

Theorem 5.2 *Assume that:*

1) *there is a regular VN $P(x)$, such that*

$$\dot{z}(t) = Mz(t) + Nz_t + q, \tag{70}$$

(with $q > 0$), is a linear time-invariant comparison system of (7);

2) *there is a vector $d > 0$ such that $(Md + N\varphi_d) < 0$.*

Then $\mathcal{A} = \{\varphi \in \mathcal{C}\colon P(\varphi(s)) \leq z_e, \forall\, s \in [-\tau, 0]\}$, where $z_e\colon Mz_e + N\varphi_{z_e} + q = 0$, is globally uniformly exponentially stable.

Example 5.2 Considering system (63) and using (64) one can obtain:

$$D_tP(x) \leq \begin{pmatrix} -5 & 1 \\ 1 & -5 \end{pmatrix} P(x) + \int\limits_{t-\tau}^{t} P(x(s))ds + 4\sqrt{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{71}$$

The associated comparison system has an equilibrium point defined by

$$z_e = \frac{4\sqrt{2}}{4-\tau}\,(1,1)^T. \tag{72}$$

Let $d = \delta(1,1)^T$. One can easily check that condition 2) of Theorem 5.2 holds if $\tau < 4$. Considering any $\delta > \frac{4\sqrt{2}}{4-\tau}$ allows us to conclude that if $\tau < 4$, then the set

$$\mathcal{A} = \left\{ \varphi \in \mathcal{C} \colon P(\varphi(s)) \leq \frac{4\sqrt{2}}{4-\tau}\,(1,1)^T,\ \forall\, s \in [-\tau, 0] \right\}$$

is uniformly exponentially stable.

Note that in the particular case $\tau = 0$ we once more obtain the results proved in [37, 39]: the set $\mathcal{A} = \{x \in \mathbb{R}^n \colon P(x) \leq \sqrt{2}(1,1)^T\}$ is uniformly exponentially stable for the corresponding ordinary differential system (69).

5.2 Positive invariance

In this part, we define the notion of positive invariant sets for general, functional systems (7) and show how this property can be investigated by means of comparison systems.

Definition 5.6 A *set* $\mathcal{M} \subset \mathcal{C}$ is said to be *positively invariant* with respect to (7) and t_0 if, for any element $\varphi \in \mathcal{M}$, the solution $x(t, t_0, \varphi)$ of (7) is defined on $[t_0, \infty)$ and $x_t(t_0, \varphi) \in \mathcal{M}$ for all $t \geq t_0$.

Let us assume that

$$\dot{z}(t) = g(t, z(t), z_t) \tag{73}$$

is a comparison system of (7) with respect to some vector Lyapunov function V. Then, the following result is obtained:

Theorem 5.3 *If there is a k-vector z_e with non-negative components such that*

$$g(t, z_e, \varphi_e) < 0, \quad \forall\, t \geq t_0,$$

where $\varphi_e \in \mathcal{C}(\mathbb{R}^k)$ is defined by $\varphi_e(s) = z_e$ for all s in $[-\tau, 0]$, then the set $\mathcal{C}(\mathcal{A})$, with $\mathcal{A} = \{x \in \mathbb{R}^n \colon V(x) < z_e\}$, is positively invariant for (7).

Proof The function g is non-decreasing in its third argument, so the system

$$\dot{y}(t) = g(t, y(t), \varphi_e) \tag{74}$$

is a comparison system of (73) as long as $y(t) \leq z_e$. System (74) is a nonlinear, ordinary differential system, and applying Lemma 6 of [39] completes the proof.

Such a vector z_e may be proved to exist in some particular cases. For instance, consider a linear comparison system of the form

$$\dot{z}(t) = M z(t) + \sum_{i=1}^{k} N^i z(t - \tau_i(t)) + \int_{-\tau}^{0} K(s)\, z(t+s)\, \mathrm{d}s.$$

If the matrix $Z = M + \sum_{i=1}^{k} N^i + \int_{-\tau}^{0} K(s)\, \mathrm{d}s$ is the opposite of an irreducible M-matrix, it is then possible to choose the vector z_e as a non-negative eigenvector associated with the importance eigenvalue of Z (i.e., the eigenvalue with the greatest real part). Proof of this result can be found in [10], and this reference also contains other results with less restrictive assumptions about the comparison system.

The positive invariance property is a standard tool in the problems of constrained control, we refer interested readers to the books [7, 43] and the papers [11, 14, 39].

6 Some Corollaries in the Ordinary Case

We recall here some results previously presented in [7], that appear as corollaries of the present study in the particular case of ordinary differential equations.

Let us consider the ordinary differential system (10) and suppose that the choice of some regular vector norm leads, for state vector in $\Omega \subset \mathbb{R}^n$, to an inequality such as

$$D^+ P(x(t)) \leq h(t, x(t), P(x(t))).$$

Several classes of functions h are investigated in the following results.

Theorem 6.1 *If it is possible to define a linear, time-invariant comparison system of (10) in a neighborhood $\Omega \subset \mathbb{R}^n$ of the equilibrium point $x = 0$ relative to a regular vector norm P:*

$$D^+ P(x(t)) \leq M\, P(x(t)), \quad \forall\, t \geq t_0, \quad \forall\, x \in \Omega$$

for which the (constant) matrix M is irreducible and of the Hurwitz type, then $x = 0$ is locally asymptotically stable and its domain of asymptotic stability $\mathcal{D}_{as}(0)$ includes the biggest domain $\mathcal{D}$ which is included in Ω and defined by one of the three following types of sets (or their union):

1. $\mathcal{D} = \mathcal{D}_1 = \{x \in \mathbb{R}^n \colon P^T(x)\, u_m(M^T) \leq \alpha\} \subset \Omega,$
2. $\mathcal{D} = \mathcal{D}_\infty = \{x \in \mathbb{R}^n \colon P(x) \leq \beta\, u_m(M)\} \subset \Omega,$
3. $\mathcal{D} = \mathcal{D}_c = \{x \in \mathbb{R}^n \colon P(x) \leq c\} \subset \Omega$, *where $c \in \mathbb{R}^k_+$ is such that $M\,c < 0$,*

in which α and β are constant scalars and $u_m(A)$ denotes a positive importance eigenvector of the M-matrix A corresponding to its importance eigenvalue $\lambda_m(A) (< 0)$.

Moreover, the convergence in this domain is exponential (in $e^{\lambda_m(M)t}$).

If $\Omega \subset \mathbb{R}^n$, the property is global.

Theorem 6.2 *If it is possible to define a nonlinear and/or time-varying comparison system of (10) in a neighborhood $\Omega \subset \mathbb{R}^n$ of the equilibrium point $x = 0$, relative to a regular vector norm P:*

$$D^+P(x(t)) \leq M(t, x(t))\, P(x(t)), \quad \forall t \geq t_0, \quad \forall x \in \Omega$$

for which the matrix $M(t, x(t))$ verifies the following properties:

1. *$M(t, x(t))$ is irreducible $\forall t \geq t_0$, $\forall x \in \Omega$;*
2. *$\exists \varepsilon > 0$ such that $M(t, x(t)) + \varepsilon I_k$ is the opposite of an M-matrix;*
3. *the non-constant elements of $M(t, x(t))$ are grouped in one single column.*

Then $x = 0$ is locally exponentially stable for (10) (in $e^{-\varepsilon t}$) and its domain of asymptotic stability $\mathcal{D}_{as}(0)$ includes the biggest domain $\mathcal{D}$ of the type:

$$\mathcal{D} = \{x \in \mathbb{R}^n : P^T(x)\, u_m(M^T) \leq \alpha\}, \quad \textit{with } \mathcal{D} \subset \Omega,$$

in which $u_m(M^T)$ is a common importance eigenvector of the matrices $M^T(t, x(t))$.

Theorem 6.3 *If it is possible to define a non-homogeneous, linear, time-invariant comparison system of (10) in the neighborhood $\Omega \subset \mathbb{R}^n$ of $x = 0$, relative to a regular vector norm P:*

$$D^+P(x(t)) \leq M\, P(x(t)) + q, \quad \forall t \geq t_0, \quad \forall x \in \Omega$$

such that

1. *M is a constant matrix of Hurwitz type,*
2. *q is a constant, non-negative vector,*
3. *the set $\mathcal{L} \triangleq \{x \in \mathbb{R}^n : P(x) \leq -M^{-1}q\}$ is strictly included in Ω.*

Then, the set $\mathcal{L}$ is asymptotically stable for system (10). Moreover, the sets $\mathcal{D}_1, \mathcal{D}_\infty$ or $\mathcal{D}_c$ (Theorem 6.1) are estimates of the domain of asymptotic stability $\mathcal{D}_{as}(\mathcal{L})$ of $\mathcal{L}$ for any scalar α or vector c such that $\mathcal{L} \subset \mathcal{D}_i \subset \Omega$ (with $i = 1, \infty$ or c).

7 Conclusion

As shown in this paper, the comparison principle method is a general and useful tool in the investigation of the asymptotic behavior of many dynamical systems. It does not replace Lyapunov's second method but it may be seen as an intermediate step allowing for the construction of a Lyapunov function. In order to be efficient, tools have to be developed which facilitate the construction of comparison systems. One of the aims of this paper is to present such a tool, developed in the framework of stability study, by our team since the 1970's. Initial applications of this

method concerned discrete or continuous nonlinear systems. The present cha extends this works to a large family of systems — the class of functional differe equations.

Using the attractive topological properties of vector norms, and with sligh strictions on the structure of the obtained comparison systems, we have st stability conditions that are easy to check (this is an important point for a p tical use of the method). In addition, some means for estimating the domair stability have been provided. Of course, there may be some cases which rec arrangements the user to make some in order to reduce the conservatism of methods. As a result, there are some works that have not been presented in chapter and which may be useful: we refer the interested reader to the pa [5, 8]. Besides, analogous results are available for discrete-time systems (see and systems of neutral type ([45]).

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