RICCATI EQUATIONS IN STABILITY THEORY OF DIFFERENCE EQUATIONS WITH MEMORY

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Abstract

This paper defines several Riccati equations that allow checking the stability of difference equations with delay effect as $x_{i+1} = \sum_{j=0}^{m} A_j x_{i-j}$ $(x_i \in \Re^n)$*. These various matrix Riccati equations have the same dimension* n *than the vector* x*, whatever the order* m *may be: this represents an advantage for high orders* m *when compared to classical matrix Lyapunov equations which should be of order* mn*. For instance, as a corollary, independent-on-delay (*m*) conditions are derived in the special case* $x_{i+1} = A x_i + Bx_{i-m}$*. All the proposed conditions are sufficient, but tend to necessary-and-sufficient ones if there is no delay effect (*$A_j = 0$ *for* $j \geq 0$*).*

1 Introduction

Many recent results are dealing with the modeling, stability and control of continuous-time, differential-difference equations (see e.g. [K-S 96, D-V 97, RIC 98] and included references). The stability of difference systems was not so often considered, even if significant applications are existing (sampling control of delay systems, or asynchronous block iterative computations [KBS 90]): in the linear case, except some recent results [V-I 95], the main tools remain the usual test of the system eigenvalues, or the classical, quadratic stability approach involving discrete Lyapunov equations (see for instance [JUR 90, K-B 60, LAS 86, B-T 96]).

Consider for instance the difference equation

$$x_{i+1} = A x_i, \quad i \geq 0, x_i \in \Re^n, \text{ A constant (n×n) matrix.} \quad (1)$$

It is well known [K-B 60] that a necessary and sufficient condition (N.S.C.) for the asymptotic stability of (1) is the existence of a symmetric, positive-definite matrix P which is the (unique) solution of the matrix Lyapunov equation

$$A'PA - P = -Q \quad (2)$$

for any symmetric, positive-definite matrix Q.

Consider now the simple difference equation with delay,

$$x_{i+1} = A x_i + Bx_{i-m}, \ i \geq 0. \quad (3)$$

The scheme (1)-(2) may again be considered for stability investigation of (3): a new phase vector $y_i = [x_i', \ldots, x_{i-m}']'$ has to be introduced, which allows rewriting (3) in the form (1) but with dimension mn. After this, stability condition in the form (2) can be used.

But, two obstacles arise in this case:

i) the dimension of y tends to infinity as m tends to infinity (it means that the dimension of the vector y and matrix P tends to infinity);
ii) if we try to investigate robust stability of (3) with respect to m (it means, independently of the value of m), then we have to check an infinite number of conditions of the form (2), (i.e. for each value of m).

Other ways to investigate stability of (1) in a necessary and sufficient manner are connected with the very classical location of the roots of the characteristic equation corresponding to (1), or to existence of an integer $m \geq n$, of a matrix $\Gamma \in \Re^{m\times n}$ with infinite norm less than one, and of a matrix $L \in \Re^{n\times m}$, such that $A'L - L\Gamma' = 0$.

However, the attempts to generalize these approaches for (3) are connected with the same obstacles (i) and (ii) mentioned above.

To overcome these computational limits, some *sufficient* conditions were derived from the use of particular norms, leading to root-location and matrix techniques (Gershgorin circles, Cassini ovals, Metzler-matrices).

Recently, a more interesting result by Verriest and Ivanov [V-I 95] really considered equation (3) as a delay system: this allowed to investigate its stability by using a Lyapunov-Krasovkii like approach, connected with Riccati matrix equations in the matrix P. This way allows keeping the n×n dimension of these matrix Riccati equations (then, n(n+1)/2 parameters) for all values of delay m, and consequently to obtain robust stability conditions with respect to m.

The present paper enlarges these results in several senses:

- we consider the more general equations, as $x_{i+1} = A_0x_i + A_1x_{i-1} \ldots + A_mx_{i-m}$ (then, including intermediary terms) and the time-varying case $x_{i+1} = A_ix_i + B_ix_{i-1}$, for instance;
- moreover, for the same original equation, we explain the way to obtain various matrix Riccati equations by introducing various transformations (or various auxiliary Lyapunov functions) and iterations.

Of course, the greater number of Riccati equations we have, the greater part of stability domain we can obtain. Besides, note that all these Riccati equations will coincide with (2) if one suppose the delay is absent.

2 Basic result

The basic tool we shall use to obtain Riccati equations is connected with the following Lyapunov-like theorem, stated below for system (3), where notation $|.|$ is a norm in $\Re^n$.

THEOREM: System (3) is asymptotically stable if there is a function $V(i, x_{-m}, ..., x_0)$ such that

$\omega_1(|x_0|) \le V(i, x_{-m}, ...,x_0) \le \omega_2(|x_{-m}| + ... + |x_0|)$

and $\Delta V(i, x_{-m}, ..., x_0) \le -\omega_3(|x_0|)$,

where ΔV denotes the shift of the function V along the solutions of:

$$\Delta V(i, x_{-m}, ..., x_0) = V(i+1, x_{-m+1}, ..., x_1 = Ax_0 + Bx_{-m}) - V(i, x_{-m}, ..., x_0), \quad (3)$$

with continuous, positive, increasing, scalar functions ω_j such that $\omega_j(0) = 0$.

3 Introductory results: the simple case m=1

Let us describe in details the proposed procedure for equations (3) with m = 1.

$$x_{i+1} = Ax_i + Bx_{i-1} \quad (4)$$

Put the Lyapunov equation V for (4) as a sum $V = V_1 + V_2$ and take $V_1 = x'_i P x_i$ where symmetric matrix $P > 0$ has to be defined. We have:

$$\Delta V_1 = x'_{i+1} P x_{i+1} - x'_i P x_i \quad (5)$$
$$= (Ax_i + Bx_{i-1})' P (Ax_i + Bx_{i-1}) - x'_i P x_i$$
$$= x'_i[A'PA - P]x_i + x'_iA'PBx_{i-1} + x'_{i-1}B'PAx_i + x'_{i-1}B'PBx_{i-1}.$$

Now let us choose V_2:

$$V_2 = x'_{i-1}[R + B'PB] x_{i-1}, \quad (6)$$

where $R > 0$ is a symmetric matrix to be chosen. Then,

$$\Delta V_2 = x'_i[R + B'PB] x_i - x'_{i-1}[R + B'PB] x_{i-1} \quad (7)$$

But, by adding and subtracting a term $x'_iA'PBR^{-1}B'PAx_i$ to some part of (5):

$$x'_iA'PBx_{i-1} + x'_{i-1}B'PAx_i - x'_{i-1}Rx_{i-1} \pm x'_iA'PBR^{-1}B'PAx_i \quad (8)$$
$$= -[Rx_{i-1} - B'PAx_i]' R^{-1}[Rx_{i-1} - B'PAx_i] + x'_iA'PBR^{-1}B'PAx_i.$$

From (5)-(8), it follows that

$$\Delta V = x'_i[A'PA - P + R + B'PB + A'PBR^{-1}B'PA]x_i - [Rx_{i-1} - B'PAx_i]' R^{-1} [Rx_{i-1} + B'PAx_i] .$$

Hence we obtain the following, first result.

THEOREM 1: *Assume that for some matrix* $R > 0$ *there exists a matrix* $P > 0$ *such that the matrix*

$$A'PA + B'PB - P + R + A'PBR^{-1}B'PA = -Q \quad (9)$$

is negative-definite. Then (4) is asymptotically stable.

Now, let us show how, by using an other type of rewriting of type (8), we could deduce a different Lyapunov candidate function and, in turn, obtain an other Riccati equation. Choose V_1 as previously: then for ΔV_1 we have representation (5). Note that, by adding and subtracting the term x'_iRx_i to some part of (5),

$$x'_iA'PBx_{i-1} + x'_{i-1}B'PAx_i \pm x'_iRx_i$$
$$= -[Rx_i - A'PBx_{i-1}]'R^{-1}[Rx_i - A'PBx_{i-1}] + x'_{i-1}B'PAR^{-1}A'PBx_{i-1} + x'_iRx_i. \quad (10)$$

Hence, if we choose as V_2 the function

$$V_2 = x'_{i-1}B'PAR^{-1}A'PBx_{i-1} + x'_{i-1}B'PBx_{i-1} , \quad (11)$$

the difference $\Delta(V_1+V_2)$ satisfies:

$$\Delta(V_1+V_2) = x'_i[A'PA-P+R+B'PB+B'PAR^{-1}A'PB]x_i - [Rx_i - A'PBx_{i-1}]'R^{-1}[Rx_i - A'PBx_{i-1}].$$

This allows deriving the following theorem.

THEOREM 2: *Assume that for some* $R > 0$ *there exists a matrix* $P > 0$ *such that the matrix*

$$A'PA - P + R + B'PB + B'PAR^{-1}A'PB = -Q \quad (12)$$

is negative-definite. Then (4) is asymptotically stable.

Comparison of (9) and (12) shows that they differ only in nonlinear (quadratic) terms where matrices A and B are replacing each other.

Remark 1: Using other transformations, it is possible to obtain conditions in terms of existence of linear matrix equation of dimension n (n(n+1)/2 parameters). In fact, take $V_1 = x_i'Px_i$ as previously and $V_2 = 2x_{i-1}'B'PBx_{i-1}$. Then:

$$\Delta(V_1+V_2) = x_i'[A'PA-P+2B'PB]x_i + x_i'A'PBx_{i-1} + x_{i-1}'B'PAx_i - x_{i-1}'B'PBx_{i-1} \pm x_i'A'PAx_i$$
$$= x_i'[2A'PA-P+2B'PB]x_i - [Bx_{i-1}+Ax_i]'P[Bx_{i-1}+Ax_i].$$

Hence, *if there exists a matrix* $P > 0$ *such that the matrix* $2(A'PA+B'PB)-P = -Q$ *is negative definite, then system (3) is asymptotically stable*. But it seems that this linear description of the matrix P is more restrictive comparing with the Riccati one because, e.g., if B = 0, the Riccati approach coincides with the necessary and sufficient condition (2), while the linear one does not.

Remark 2: Other possibilities to obtain various Riccati equations arise if we iterate right hand side of (3) several times. We shall describe this approach in more details after consideration of the general case.

4 The general case

Consider now equation

$$x_{i+1} = \sum_{j=0}^{m} A_j x_{i-j} \quad (13)$$

($i \ge 0$, $A_{j\ (j=0,1,2,...,m)}$ constant $n \times n$ matrices, m fixed integer).

Similarly to (9), we obtain the following theorem (in the following equation (14), it is assumed that the sum is equal to zero if the upper limit of the summation is less than the lower one).

THEOREM 3: *Assume that for some matrices* $R_i > 0$ *there exists a matrix* $P > 0$ *such that the matrix* (14)

$$\sum_{j=0}^{m} A_j'PA_j - P + \sum_{j=0}^{m} R_j + \sum_{j=0}^{m-1} A_j'P \sum_{l=j+1}^{m} A_lR_l^{-1}A_l'PA_j = -Q < 0$$

is negative definite. Then, the system (13) is asymptotically stable.

Other Riccati equations (like (12) for the case m=1*) can be obtained from (14) if in any arbitrary chosen nonlinear term, or any set of terms*

$$A_j'PA_lR_l^{-1}A_l'PA_j , \quad (15)$$

the matrices A_j *and* A_l *are formally replaced by the places, which means, e.g., that instead of (15), we have the term*

$$A_l'PA_jR_l^{-1}A_j'PA_l . \quad (16)$$

Proof. *First part of Theorem 3:* for the shake of simplicity, let us consider the case m = 2,

$$x_{i+1} = A_0\, x_i + A_1\, x_{i-1} + A_2\, x_{i-2}\,. \qquad (17)$$

Take $V_1 = x'_i P x_i$. Then

$$\Delta V_1 = (A_0\, x_i + A_2\, x_{i-2})'P(A_0\, x_i + A_2\, x_{i-2})$$
$$+x_{i-1}'A_1'P\,(A_0\, x_i + A_2\, x_{i-2})+(A_0\, x_i + A_2\, x_{i-2})'PA_1x_{i-1}$$
$$+ x_{i-1}'A_1'PA_1x_{i-1} - xi'Pxi \qquad (18)$$

Take as a second part of V the function

$V_2 = x_{i-2}'[A_2'PA_2+R_1]x_{i-2}$. Then we have:

$$(A_0\, x_i + A_2\, x_{i-2})'P(A_0\, x_i + A_2\, x_{i-2})-x_i'Px_i+\Delta V_2$$
$$= x_i'(A_0'PA_0-P)x_i+ x_{i-1}'(A_2'PA_2+R_1)x_{i-1}$$
$$- (R_1x_{i-2}-A_2'PA_0x_i)'R_1^{-1}(R_1x_{i-2}-A_2'PA_0x_i) \qquad (19)$$
$$+ x_i'A_0'PA_2R_1^{-1}A_2'PA_0x_i$$

Take now as third part $V_3=x_{i-1}'(A_2'PA_2+ A_1'PA_1+R_1+R_2)x_{i-1}$. Then,

$$x_i'A_0'PA_1x_{i-1}+x_{i-1}'A_1'PA_0x_i$$
$$+ x_{i-1}'(A_2'PA_2+A_1'PA_1+R_1)x_{i-1}+\Delta V_3$$
$$= x_i'(A_2'PA_2+ A_1'PA_1+R_1+R_2)x_i$$
$$+ x_i'A_0'PA_1R_2^{-1}A_1'PA_0x_i$$
$$- (R_2x_{i-1}-A_1'PA_0x_i)'R_2^{-1}(R_2x_{i-1} - A_1'PA_0x_i)\,. \qquad (20)$$

At last, put $V_4 = x_{i-1}'R_3x_{i-1}$. Then:

$$x_{i-1}'A_1'PA_2x_{i-2} + x_{i-2}'A_2'PA_1x_{i-1} +\Delta V_4$$
$$= x_i'R_3x_i - [R_3x_{i-1}-A_1'PA_2x_{i-2}]'R_3^{-1}[R_3x_{i-1}-A_1'PA_2x_{i-2}]$$
$$+ x_{i-2}'A_2'PA_1R_3^{-1}A_1'PA_2x_{i-2}\,. \qquad (21)$$

So, if we take $V = V_1 + V_2 + V_3 + V_4 + V_5 + V_6$, where

$V_5 = x_{i-2}'\, A_2'PA_1R_3^{-1}A_1'PA_2\, x_{i-2}$

and $V_6 = x_{i-1}'A_2'PA_1R_3^{-1}A_1'PA_2x_{i-1}$,

we obtain by virtue of (18)-(22):

$$\Delta V= x_i'[-P+\sum_{j=0}^{2}\ (A_j'PA_j+R_j) + A_0'PA_2R_1^{-1}A_2'PA_0$$
$$+ A_0'PA_1R_2^{-1}A_1'PA_0 + A_2'PA_1R_3^{-1}A_1'PA_2]x_i$$
$$- (R_1x_{i-2}-A_2'PA_0x_i)'R_1^{-1}(R_1x_{i-2}-A_2'PA_0x_i)$$
$$- (R_2x_{i-1}-A_1'PA_0x_i)'R_2^{-1}(R_2x_{i-1}-A_1'PA_0x_i)$$
$$- (\, R_3x_{i-1}-A_1'PA_2x_{i-2})'R_3^{-1}(R_3x_{i-1}-A_1'PA_2x_{i-2}). \qquad (23)$$

We obtain from (23) one of the possible Riccati equations :

$$\sum_{j=0}^{2}\ (A_j'PA_j+R_j) + A_0'PA_2R_1^{-1}A_2'PA_0$$
$$+ A_0'PA_1R_2^{-1}A_1'PA_0 + A_2'PA_1R_3^{-1}A_1'PA_2 - P = -Q\,. \qquad (24)$$

Second part of Theorem 3: other Riccati equations can be obtained from (24) by changing matrices A_i and A_l or some nonlinear terms. Let us justify the possibility of one of the interchanges for the last summand in the left-hand side of (24) (other changes are justified just in the same manner). Let us keep without any change the transformations (18)-(20) and make some modifications beginning with the formula (21). Namely, put $V_4 = x_{i-2}'R_3x_{i-2}$. Then, instead of (21) we obtain:

$$x_{i-1}'A_1'PA_2x_{i-2} + x_{i-2}'A_2'PA_1x_{i-1} + \Delta V_4$$
$$= - [R_3x_{i-2}-A_2'PA_1x_{i-1}]'R_3^{-1}[R_3x_{i-2}-A_2'PA_1x_{i-1}]$$
$$+ x_{i-1}'[R_3+A_1'PA_2R_3^{-1}A_2'PA_1]x_{i-1} \qquad (25)$$

From (25), it is clear that we must choose as V_5

$$V_5 = x_{i-1}'[R_3+A_1'PA_2R_3^{-1}A_2'PA_1]x_{i-1}\,. \qquad (26)$$

From (18)-(20), (25) and (26), it follows that:

$$\Delta(V_1+...+V_5) = x_i'[-P+\sum_{j=0}^{2}(A_jPA_j+R_j)(A_jPA_j+R_j)$$
$$+ A_0'PA_2R_1^{-1}A_2'PA_0 + A_0'PA_1R_2^{-1}A_1'PA_0$$
$$+A_1'PA_2R_3^{-1}A_2'PA_1]x_i$$
$$-(R_1x_{i-2}-A_2'PA_0x_i)'R_1^{-1}(R_1x_{i-2}-A_2'PA_0x_i)$$
$$- (R_2x_{i-1}-A_1'PA_0x_i)'R_2^{-1}(R_2x_{i-1}-A_1'PA_0x_i)$$
$$- (R_3x_{i-2}-A_2'PA_1x_{i-1})'R_3^{-1}(R_3x_{i-2}-A_2'PA_1x_{i-1}).$$

As a result, we obtain Riccati equation for P:

$$\sum_{j=0}^{2}\ (A_j'PA_j+R_j) + A_0'PA_2R_1^{-1}A_2'PA_0$$
$$+ A_0'PA_1R_2^{-1}A_1'PA_0+A_1'PA_2R_3^{-1}A_2'PA_1-P = -Q\,.$$

In a similar way, we can prove the validity of other Riccati equations for the matrix P, obtained from (24) by formal interchange of A_j and A_l , $l \le j$, in one or some terms. ■

5 Delay-independent stability conditions

Consider equation (3) in which delay m is an unknown integer. In this case, the Riccati equation corresponding to (14) is reduced to:

$$A'PA + B'PB + R - P + A'PBR^{-1}B'PA = -Q\,, \qquad (27)$$

and the analogous one, using (15)-(16), is

$$A'PA + B'PB + R - P + B'PAR^{-1}A'PB = -Q\,. \qquad (28)$$

Both of these equations do not depend on m and, hence, give us asymptotic stability conditions for all values of $m > 0$, as stated in the following corollary.

<u>COROLLARY 1</u>: *The system (3),* $x_{i+1} = A\, x_i + Bx_{i-m}$ *, is asymptotically stable independently of the delay* m *if, for some* $Q > 0$ *and* $R > 0$*, there exists a matrix* $P > 0$ *satisfying one of the Riccati equations (27) or (28).*

Remark that this results was also given in [V-I 95].

Example: Let equation (3) be a scalar one:

$$x_{i+1} = ax_i + bx_{i-m}\,. \qquad (29)$$

Then, both equation (27) and (28) being applied to (29) coincide and have the form

$$(a^2 + b^2 -1 + Pb^2a^2R^{-1})P + R = -Q\,. \qquad (30)$$

Let us choose R in such a way that the left-hand side of (30) is minimized with respect to $R > 0$, which means $R = P\,|ab|$. As a result, we obtain :

$$(a^2 + b^2 -1 + Pb^2a^2R^{-1})P + R = [\ (|a| + |b|)^2 -1\ ]\ P.$$

Hence, asymptotic stability condition of (29) independent on m has the form

$$|a| + |b| < 1 \qquad (31)$$

6 Iteration of the equation

Other forms of Riccati equation (and consequently, other stability conditions in the space of parameters) can be obtained if we transform some of the terms in the right-hand side of the original, difference equation. For instance, consider the simple equation (4): in the right-hand side of this equation, the terms Ax_i and Bx_{i-1} can be transformed, for example, as follows:

$$Ax_i = A(Ax_{i-1}+Bx_{i-2}), \qquad (32)$$
$$Bx_{i-1} = B(Ax_{i-2}+Bx_{i-3}),\quad \ldots \text{ and so on.}$$

Let us choose one of these transformations and rewrite equation (4) in the form

$$x_{i+1} = (A^2+B)x_{i-1}+ABx_{i-2}. \qquad (33)$$

Then, by virtue of (14) and (33), we obtain that equation (4) is asymptotically stable if there exists the solution P > 0 of the Riccati equation:

$$B'A'PAB + (A^2+B)'P(A^2+B) + R - P + (A^2+B)'PABR^{-1}B'A'P(A^2+B) = -Q, \quad (34)$$

where R>0, Q>0 are some matrices which can be chosen arbitrarily.

Example: Consider again the scalar equation (29), now with m=1. Of course, necessary and sufficient conditions of asymptotic stability, in this case, are known to be

$$|b| < 1 \quad \text{and} \quad |a| < 1-b \quad (35)$$

However, as an illustration, let us apply the above procedure: the equation (34) gives us:

$$[(a^2+b)^2+b^2a^2 -1 + R^{-1}(a^2+b)^2b^2a^2P]P + R = -Q$$

The point $R_0 > 0$ where the left-hand side will be minimum is $R_0 = P|(a^2+b)ab|$. At this point,

$$[(a^2+b)^2 + b^2a^2 - 1 + R_0^{-1}(a^2+b)^2b^2a^2P]P + R_0 = [(|a^2+b| + |ab|)^2 - 1]P.$$

Hence, using this approach, we obtain the following stability condition of the scalar equation (29) with m = 1 :

$$|a^2+b|+|ab| < 1. \quad (36)$$

Figure 1 shows that for b ≥ 0, inequality (36) represents the same part of stability domain as (31); but for b < 0, it provides (in the case m = 1) a greater estimate of this stability domain.

We can do further approximation and rewrite equation (29) in the form:

$$x_{i+1} = (a^3 + 2ab)\, x_{i-2} + b(a^2 + b)\, x_{i-3}\,. \quad (37)$$

Riccati equation for (37) by virtue of (14) has a form

$$(a^3+2ab)^2 + b^2(a^2 + b)^2 + R + (a^3 + 2ab)^2 b^2 (a^2 + b)^2 R^{-1} P = -Q$$

Choosing here R > 0, as previously from the condition of the minimum of the left-hand side, we obtain that the stability domain is defined by the inequality:

$$|a^3+2ab| + |b(a^2+b)| < 1 \quad (38)$$

Inequality (38) gives, for b > 0, the same part of stability domain as (31) but, for b < 0, represents the greater part of it comparing with (31) and also to (36).

8 Other, various Lyapunov functions

8.1 First variation

So far as, the Lyapunov function V_1 was chosen in a simple quadratic form $V_1 = x_i'Px_i$. But, of course, as V_1 can be taken other suitable functions which could lead us to other Lyapunov functionals V and as a result to the greater part of stability domain in the space of parameters. Let us illustrate this kind of action by considering equation (4). Take as a function V_1 the following:

$$V_1 = (x_i + Bx_{i-1})'\, P\, (x_i + Bx_{i-1}),$$

where matrix P = P' > 0 has to be defined. We then have:

$$\Delta V_1 = (x_{i+1} + Bx_i)'\, P\, (x_{i+1} + Bx_i) - (x_i + Bx_{i-1})'\, P\, (x_i + Bx_{i-1})$$
$$= (Cx_i + Bx_{i-1})'\, P\, (Cx_i + Bx_{i-1}) - (x_i + Bx_{i-1})'P\, (x_i + Bx_{i-1}),$$

where C = A+B. Now let us take $V_2 = x_{i-1}'Rx_{i-1}$, where R > 0.

Then, I being the identity matrix, we have for $V=V_1+V_2$:

$$\Delta V = x_i'[C'PC-P+R]x_i + x_i'[C'-I]PBx_{i-1} + x_{i-1}'B'P[C-I]x_i - x_{i-1}'Rx_{i-1}$$
$$= x_i'[C'PC-P+R+(C-I)'PBR^{-1}B'P(C-I)]x_i - [Rx_{i-1}-B'P(C-I)x_i]'R^{-1}[Rx_{i-1}-B'P(C-I)x_i],$$

which leads to the following result.

<u>THEOREM 4</u>: *Assume that for some symmetric matrices* R > 0, Q > 0*, there exists a symmetric matrix* P > 0 *satisfying the Riccati equation* (39)

$$(A+B)'P(A+B) - P + R + (A'+B'-I)PBR^{-1}B'P(A+B-I) = -Q.$$

Then system (4) is asymptotically stable.

Example: Consider once more scalar equation (29) with m=1. Then, equation (39) gives us the equation for P:

$$[(a+b)^2 -1]P + R + (a+b-1)^2 b^2 P^2 R^{-1} = -Q \quad (40)$$

Choose R from the condition minimizing the left hand side of (40) with respect of R > 0, i.e.,

$$R = P\,|(a+b-1)b|\,.$$

Then equation (40) gives us the following conditions of asymptotic stability of (29) with m=1,

$$(a + b)^2 + 2\,|b(a + b -1)| < 1\,. \quad (41)$$

Comparing with the above obtained stability domain, inequality (41) represents an improvement: stability domain given by (41) is shown on Fig. 1 which shows, in particular, that system (29) can be asymptotically stable even for a > 1 . It means that stability, in this case, is achieved due to the presence of delay in equation (29).

8.2 Second variation

Using other Lyapunov functions V_1, we should be able to enlarge stability domain. Consider once more equation (4) and take as the Lyapunov function V_1 the following:

$$V_1 = (x_i, x_{i-1}) \begin{bmatrix} P_1 & P_3 \\ P_3' & P_2 \end{bmatrix} (x_i, x_{i-1})' \quad (42)$$

where P_1 and P_2 are symmetric positive definite (n×n) matrices, matrix P_3 is (n×n) and the choice of these matrices must be done in such a way that the matrix P,

$$P = \begin{bmatrix} P_1 & P_3 \\ P_3' & P_2 \end{bmatrix} \quad (43)$$

is symmetric and the quadratic form (42) is positive definite with respect to either x_i or x_{i-1}.

Let us introduce also $V_2 = x_{i-1}'\, R\, x_{i-1}$, where R > 0 . Then, for $V = V_1+V_2$, we have:

$$\Delta V = x_i'[A'P_1A + A'P_3 + P_3'A + P_2 - P_1 + R \quad (44)$$
$$+(B'P_1A+B'P_3-P_3)'R^{-1}(B'P_1A+B'P_3-P_3)]x_i + x_{i-1}'(B'P_1BP_2)x_{i-1}$$
$$-[Rx_{i-1}-(B'P_1A+B'P_3-P_3')x_i]'R^{-1}[Rx_{i-1}-(B'P_1A+B'P_3-P_3')x_i].$$

Let us look for a matrix P satisfying the above-formulated assumptions in the form:

$$P_3 = CP_1\,,\ P_2 = I+B'P_1B\,, \quad (45)$$

where C_1 is some non-negative constant matrix and I is the identity matrix.

From (44)(45) the following result follows.

<u>THEOREM 5</u>: *Assume that for some symmetric matrices* R > 0, Q > 0, C > 0, *there exists a symmetric matrix* $P_1 > 0$ *satisfying the Riccati equation*

$$A'P_1A + C(A'P_1+P_1A) + I + B'P_1B-P_1 + R + (B'P_1A+CB'P_1-CP_1)'R^{-1}(B'P_1A+CB'P_1-CP_1) = -Q. \quad (46)$$

Then, system (4) is asymptotically stable.

Example: Let us apply this approach to the scalar equation:
$x_{i+1} = ax_i + bx_{i-1}$.
Then, from (45) and (46), it follows that:

$$P_1 = 2(1-b)(1-b)^{-1}[(1-b)^2-a^2]^{-1} ,$$
$$P_2 = 1+b^2P_1 , \quad (47)$$
$$P_3 = ab(1-B)^{-1}P_1 .$$

The matrix (43) with the entries (47) is positive definite if and only if:

$$|b| < 1 \text{ and } |a| < 1-b . \quad (48)$$

Hence, the last approach has given us the whole, necessary and sufficient stability domain (35) in the space of parameters (a, b) (see Fig. 1).

9 Time-dependent systems

The same description of stability properties in terms of the solutions of Riccati equations can be also done for some discrete equations with time-varying parameters. Necessary and sufficient conditions were obtained [OPO 86] for :

$$x_{i+1} = A_i x_i \ , \quad i \geq i_0 \quad (49)$$

with A_i belonging to some compact set.
These conditions (existence of a norm | . | over $\Re^n$ such that for any i, $|A_i x_i| < \alpha |x_i|$ with some unique real α belonging to [0,1[) are not so easy to apply (the problem is to find this norm) and, *a fortiori*, encounter the same computational limits when considering delayed equations as (3) (here, with varying A and B). This section will apply the same idea as previously. For simplicity sake, we shall consider here the case m=1, but generalization to any m is possible. Then, consider the equation:

$$x_{i+1} = A_i x_i + B_i x_{i-1} \ , \ i \geq i_0 , \quad (50)$$

where $x_i \in \Re^n$ and A_i, B_i are some prescribed (n×n) matrices. Let us take $V_1 = x_i' P_i x_i$, then:

$$\Delta V_1 = (A_i x_i + B_i x_{i-1})' P_{i+1} (A_i x_i + B_i x_{i-1}) - x_i' P x_i$$
$$= x_i'[A_i' P_{i+1} A_i - P_i] x_i + x_i' A_i' P_{i+1} B_i x_{i-1}$$
$$+ x_{i-1}' B_i' P_{i+1} A_i x_i + x_{i-1}' B_i' P_{i+1} B_i x_{i-1} .$$

Let R_i be a sequence of symmetric positive definite matrices, and $V_2 = x_{i-1}'[R_i + B_i' P_{i+1} B_i] x_{i-1}$:

$$\Delta V_2 = x_i'[R_{i+1} + B_{i+1}' P_{i+2} B_{i+1}] x_i - x_{i-1}'[R_i + B_i' P_{i+1} B_i] x_{i-1} .$$

Remark also that:

$$x_i' A_i' P_{i+1} B_i x_{i-1} + x_{i-1}' B_i' P_{i+1} B_i x_{i-1} - x_{i-1} R_i x_{i-1}$$
$$= -[R_i x_{i-1} - B_i P_{i+1} A_i x_i]' R_i^{-1} [R_i x_{i-1} - B_i P_{i+1} A_i x_i]$$
$$+ x_i' A_i' P_{i+1} B_i' R_i^{-1} B_i P_{i+1} A_i x_i .$$

As a result, we obtain the following conclusion.

THEOREM 6: *If for some matrices* $R_i > 0$ *there exist symmetric matrices* $P_i > 0$ *such that all the matrices*

$$A_i' P_{i+1} A_i - P_i + R_{i+1} + B_{i+1}' P_{i+2} B_{i+1} + A_i' P_{i+1} B_i' R_i^{-1} B_i P_{i+1} A_i = Q_i$$

are uniformly negative definite, then system (50) is uniformly asymptotically stable with respect to the initial instant i_0.

10 Conclusion

Several Riccati equations have been formulated, giving sufficient stability conditions. The lack of necessity is compensated by an important reduction of the order of the involved matrices equations. As in [V-I 95], the presented Riccati-equation-based criteria, using different ways of re-writing some initial equations, can be compared to recent results presented for differential-difference systems. The original contribution can be summarized in to main points:

1) The class of difference systems considered in [V-I 95] (system (3), Corollary 1) has been enlarged to systems with several delays (system (13), Theorem 3) and time-dependent systems (system (50), Theorem 5).
2) Several ways of generalization have been presented: our aim was not only to obtain some stability conditions in the form of Riccati equations, but also to indicate how various other forms of Riccati equations can be obtained, and demonstrated some of them.
3) Moreover our approach allows using degenerate functionals and investigate the possibility of stabilization by using delay effect.

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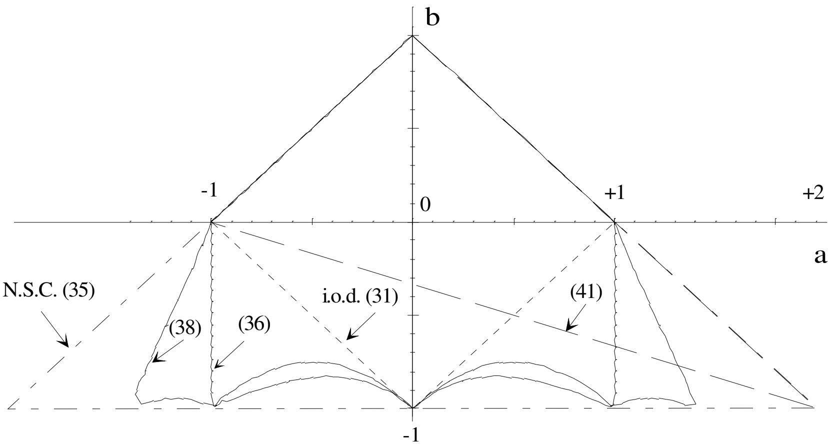

Figure 1: comparison of the stability domains (31) (delay-independent), (36), (38), (41) and the N.S.C. (35).