Contribution to the control of nonlinear systems under aperiodic sampling

Hassan Omran

Thesis supervised by:

Jean-Pierre Richard — *Professor, Ecole Centrale de Lille*
Françoise Lamnabhi-Lagarrigue — *CNRS Research Director, L2S*
Laurentiu Hetel — *CNRS Research Associate, LAGIS*

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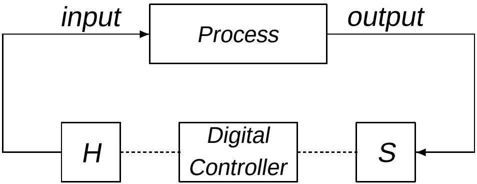
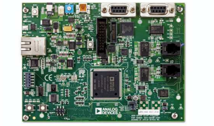
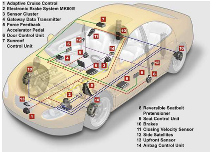
Sampled-data systems

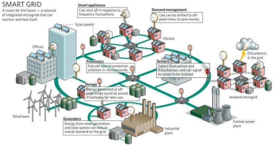




Digital/Networked Control Systems

Applications of sampled-data





The HYCON2 Project

Highly-complex and networked control systems (HYCON2)

- *FP7 project coordinated by Françoise Lamnabhi-Lagarrigue (CNRS) .*
- *WP2 is related to research on Networked Control Systems.*

Challenges in sampled-data control

Processor: *limited calculation power*
Network: *finite bandwidth*
Sampler: *minimum responding time*

$\Rightarrow$ *finite number of samples per time unit*

How fast SHOULD we sample? $\leftrightarrow$ *How fast CAN we sample?*

Challenges in sampled-data control

Sampler clock: *jitter*
Network: *packet dropouts*
Scheduling: *interaction between algorithms*
Real-time computing: *microprocessor latency*

$\Rightarrow$ *sampling is not necessarily periodic*

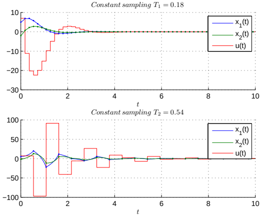
How to ensure robustness with respect to asynchronous sampling?

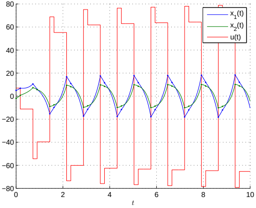
Challenges in sampled-data control

Aperiodic sampling may cause instability

$$\dot{x}(t) = A_0 x(t) + B_0 u(t), \qquad u(t) = Kx(t_k)$$

$$A_0 = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}, \quad B_0 = \begin{bmatrix} 1 \\ 0.6 \end{bmatrix}, \quad K = \begin{bmatrix} -1 & -6 \end{bmatrix}.$$

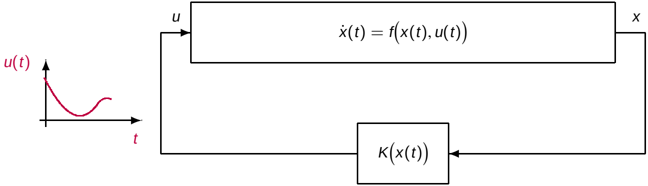




Periodic sampling $T_1 = 0.18$, $T_2 = 0.54$ *Aperiodic sampling* $T_1 \to T_2 \to T_1 \cdots$

Problem under study

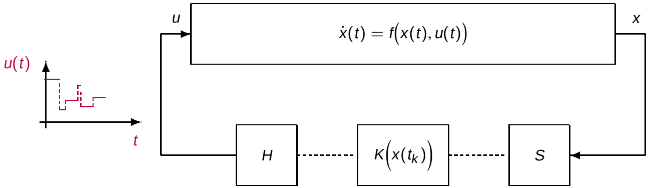
Continuous-time controller



$$u(t) = K\big(x(t)\big)$$

Problem under study

Digital implementation under asynchronous sampling (emulation approach)



$$u(t) = K\Big(x(t_k)\Big), \quad \forall t \in [t_k, t_{k+1}), \qquad 0 < \epsilon \leq t_{k+1} - t_k \leq \underbrace{h_{max}}_{MASP}, \quad \forall k \in \mathbb{N}.$$

Find stability criteria for nonlinear sampled-data control systems, which provide a computable estimate of the Maximum Allowable Sampling Period (MASP).

Existing results

The linear time-invariant case:* *Realistic model?

- *Input delay approach (Fridman et al 2004), (Fridman 2010), (Michiels 2005)*
- *Robust control based analysis (Mirkin 2007), (Fujioka 2009)*
- *Impulsive modelling (Naghshtabrizi 2008)*
- *Discrete-time approaches & convex embedding (Hetel, Daafouz et al 2007)*
- *Sum of squares (Seuret 2011)*

$\longrightarrow$ ***Bilinear case***

The nonlinear case:* *Constructive?

- *Input delay approach (Mazenc et al 2013)*
- *Hybrid system modelling (Nešić et al 2009), (Burlion et al 2006)*
- *Single/vector Lyapunov functions (Karafyllis et al 2007)*
- L_p *stability (Zaccarian et al 2003)*

Existing results

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Outline

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Bilinear systems

$$\dot{x}(t) = A_0 x(t) + \sum_{i=1}^{m} [u(t)]_i N_i x(t) + B_0 u(t)$$

- *The simplest class of nonlinear systems.*
- *Offer a more accurate approximation of nonlinear systems than the classical linear ones.*
- *Allows to address the problem for a simple nonlinear system.*
- *Several applications: power electronics, mechanical systems, chemical processes.*
- *Continuous-time stabilization techniques: quadratic; division; sliding and **linear state feedback**.*

Bilinear systems

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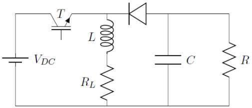
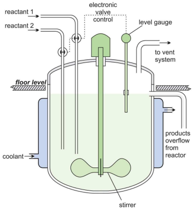
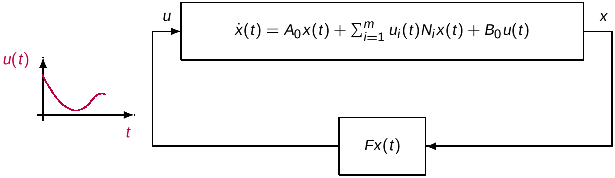




Problem formulation

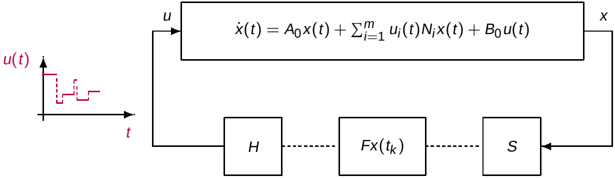
Continuous-time control of bilinear systems with linear state feedback (local stabilization):



LMI based synthesis (Amato et al 2007), (Tarbouriech et al 2009).

Problem formulation

Sampled-data implementation (emulation approach):



$$u(t) = Fx(t_k), \quad \forall t \in [t_k, t_{k+1}),$$

$$0 < \epsilon \leq t_{k+1} - t_k \leq \underbrace{h_{max}}_{MASP}, \quad \forall k \in \mathbb{N}$$

***Problem**: Constructive method to estimate the MASP, and the domain of attraction using LMI.*

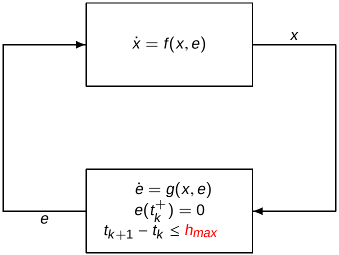
Hybrid system model

Hybrid system model for Networked Control Systems: (Walsh et al 2002) and (Nešić et al 2004) .

$\underline{e(t) = x(t_k) - x(t)}$

$$f(x,e) = \tilde{A}[x(t), e(t)]x(t) + Be(t),$$
$$g(x,e) = -\tilde{A}[x(t), e(t)]x(t) - Be(t),$$

$$\tilde{A}[x,e] := A[x(t_k)] = A_0 + B_0 F + \sum_{i=1}^{m} [Fx(t_k)]_i N_i,$$
$$B = B_0 F.$$



Hybrid system model

$$\left.\begin{aligned} \dot{x} &= f(x,e) = \tilde{A}[x,e]x + Be \\ \dot{e} &= g(x,e) = -\tilde{A}[x,e]x - Be \\ \dot{\tau} &= 1 \end{aligned}\right\} \qquad \tau \in [0, h_{max})$$

$$\left.\begin{aligned} x^+ &= x \\ e^+ &= 0 \\ \tau^+ &= 0 \end{aligned}\right\} \qquad \tau \in [\epsilon, h_{max}]$$

Hybrid system model:

$$\begin{aligned} \xi &:= [x^T, e^T, \tau]^T \\ F(\xi) &:= [f(x,e)^T, g(x,e)^T, 1]^T \\ G(\xi) &:= [x, 0, 0]^T \\ C &:= \{\xi : \tau \in [0, h_{max})\} \\ D &:= \{\xi : \tau \in [\epsilon, h_{max}]\} \end{aligned}$$

$$\begin{cases} \dot{\xi} = F(\xi), & \xi \in C, \\ \xi^+ = G(\xi), & \xi \in D. \end{cases}$$

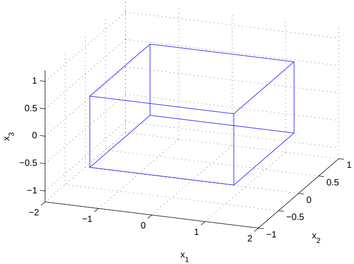
Proposed solutions

Local Analysis:

$$\mathcal{P} = conv\{x_1, x_2, \ldots, x_p\}$$

$$A[x(t_k)] \in \mathcal{P}_A = conv\{A_1, A_2, \ldots, A_p\}$$

$$A_i = A_0 + B_0F + \sum_{j=1}^{m} [Fx_i]_j N_j, \quad \forall i \in \{1, 2, \ldots, p\}$$



Method 1

Result for the general nonlinear case (Nešić et al 2009):

$$\text{Lyapunov function } U(\xi) = V(x) + \gamma\phi(\tau)W^2(e)$$

$$\begin{aligned} \left\langle \frac{\partial W(e)}{\partial e}, g(x,e) \right\rangle &\leq LW + H(x,e) \\ \langle \nabla V(x), f(x,e) \rangle &< -\varrho(|x|) - \varrho(W(e)) - H^2(x,e) + \gamma^2 W^2(e) \end{aligned}$$

$$\Rightarrow \langle \nabla U(\xi), F(\xi) \rangle < -\varrho(|x|) - \varrho(W(e))$$

***Question**: How to find $V(\cdot)$,$W(\cdot)$, $H(\cdot,\cdot)$, $\varrho(\cdot)$, γ and L? It is stated in (Nešić et al 2009) that:*

"We note that finding these functions may be hard for general nonlinear systems".

Method 1

Result for the general nonlinear case (Nešić et al 2009):

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Question*: How to find $V(\cdot)$,$W(\cdot)$, $H(\cdot,\cdot)$, $\varrho(\cdot)$, γ and L? It is stated in (Nešić et al 2009) that:*

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Method 1

Theorem (CDC'12)

Assume that the MASP is strictly bounded $h_{max} < \mathcal{T}(L,\gamma)$:

$$\mathcal{T}(L,\gamma) := \begin{cases} \arctan(r)/(Lr) & \gamma > L \\ 1/L & \gamma = L \\ \text{arctanh}(r)/(Lr) & \gamma < L \end{cases} \qquad r = \sqrt{\left|\frac{\gamma^2}{L^2} - 1\right|}$$

where L is given by

$$L = \frac{1}{2}\max\{-\lambda_{min}(B^T + B), 0\}$$

and γ is the solution to the the optimization problem $\gamma = \min \gamma'$ under the constraints: $\exists P \in \mathbb{R}^n$, $P > 0$, and $\alpha > 0$ such that

$$M_{ij} = \begin{bmatrix} A_i^T P + PA_i + \frac{1}{2}(A_i^T A_j + A_j^T A_i) + \alpha I & PB \\ * & (\alpha - \gamma'^2)I \end{bmatrix} < 0, \qquad \forall i,j \in \{1,2,...,p\}.$$

Then the bilinear sampled-data system is locally uniformly asymptotically stable.

Method 2

Method 1: Conservatism due to upper estimations of the derivative of a Lyapunov function.

Consider studying directly the Lyapunov function:

$$U'(\xi) = x^T P x + \exp\left(\frac{-\tau}{h_{max}}\right) e^T Q e$$

Theorem (CDC'12)

Assume that there exist symmetric positive definite matrices $P, Q, X, Y \in \mathbb{R}^{n\times n}$, such that the following LMIs are satisfied

$$\begin{bmatrix} A_i^T P + PA_i + X & PB - A_i^T Q \\ * & -B^T Q - QB - \frac{1}{h_{max}} Q + Y \end{bmatrix} < 0, \qquad \forall i \in \{1, 2, ..., p\}.$$

$$\begin{bmatrix} A_i^T P + PA_i + X & PB - A_i^T Q \exp(-1) \\ * & [-B^T Q - QB - \frac{1}{h_{max}} Q] \exp(-1) + Y \end{bmatrix} < 0, \qquad \forall i \in \{1, 2, ..., p\}.$$

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Then the bilinear sampled-data system is locally uniformly asymptotically stable.

Numerical example

Consider the example from (Amato et al 2007) and (Tarbouriech et al 2009):

$$A_0 = \begin{bmatrix} -0.5 & 1.5 & 4 \\ 4.3 & 6.0 & 5.0 \\ 3.2 & 6.8 & 7.2 \end{bmatrix}; \quad B_0 = \begin{bmatrix} -0.7 & -1.3 \\ 0 & -4.3 \\ 0.8 & -1.5 \end{bmatrix}$$

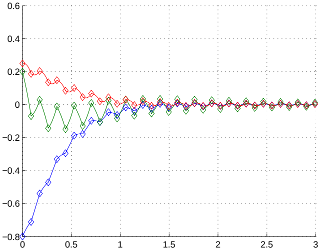
$$N_1 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}; \quad N_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

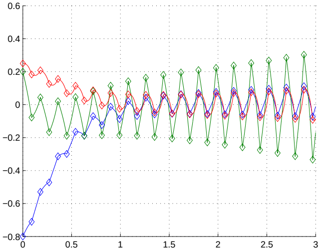
$$F = \begin{bmatrix} 0.0016 & 0.0035 & 0.0034 \\ 2.2404 & 3.2676 & 5.9199 \end{bmatrix}$$

$$\mathcal{P} = [-1.35, +1.35] \times [-0.5, +0.5] \times [-0.5, +0.5]$$

Numerical example

	Method 1 (CDC'12)	*Method 2 (CDC'12)*
h_{max}	5×10^{-3}	13×10^{-3}





$t_{k+1} - t_k = 88 \times 10^{-3}$

$t_{k+1} - t_k = 90 \times 10^{-3}$

Summary

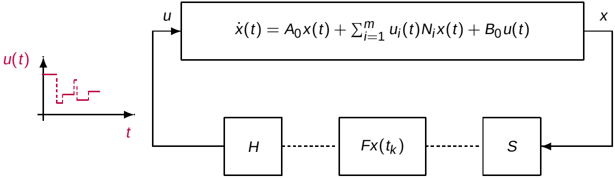
- *Stability of bilinear sampled-data systems was addressed using a hybrid systems approach.*
- ***Method 1*** *: a constructive method to apply results from (Nešić et al 2009) for the bilinear case.*

  Method 2 *: a direct search of a Lyapunov function for the hybrid system.*
- *Both methods are constructive via LMIs.*
- *Still room for improvements.*

Outline

Problem formulation

Sampled-data implementation (emulation approach):



$$u(t) = Fx(t_k), \quad \forall t \in [t_k, t_{k+1}),$$

$$0 < \epsilon \leq t_{k+1} - t_k \leq \underbrace{h_{max}}_{MASP}, \quad \forall k \in \mathbb{N}$$

Dissipativity-based representation

Equivalent model

Closed-loop system

$$\dot{x}(t) = \Big[A_0 + \sum_{i=1}^{m} (Fx(t_k))_i N_i\Big] x(t) + B_0 F x(t_k)$$

$$\dot{x}(t) = \underbrace{\Big[A_0 + B_0 F + \sum_{i=1}^{m} (Fx(t_k))_i N_i\Big]}_{\tilde{A}[x,e] := A[x(t_k)]} x(t) + \underbrace{B_0 F}_{B} \underbrace{(x(t_k) - x(t))}_{e(t)}$$

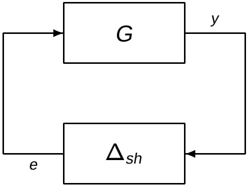
The system can be represented by the interconnection of:

$$G := \begin{cases} \dot{x}(t) = & \tilde{A}[x,e]x(t) + Be(t) \\ y(t) = & \dot{x}(t) \end{cases}$$

with the operator $\Delta_{sh} : y \to e$ defined by:

$$e(t) = -\int_{t_k}^{t} y(\tau) d\tau := (\Delta_{sh} y)(t), \quad \forall t \in [t_k, t_{k+1})$$

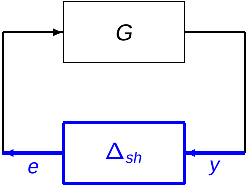
Dissipativity-based representation



$$G := \begin{cases} \dot{x}(t) = & \tilde{A}[x,e]x(t) + Be(t) \\ y(t) = & \dot{x}(t) \end{cases}$$

$$(\Delta_{sh}y)(t) := -\int_{t_k}^{t} y(\tau)d\tau.$$

Properties of the operator



L_2-induced norm (Mirkin 2007):

$$\frac{||e||}{||y||} \le \delta_0 := \frac{2}{\pi} h_{max} \quad i.e. \int_0^\infty e^T(\tau)e(\tau)d\tau \le \delta_0^2 \int_0^\infty y^T(\tau)y(\tau)d\tau.$$

Passivity (Fujioka 2009):

$$\langle \Delta_{sh} y, y \rangle = \int_0^\infty y^T(\tau)\left(\Delta_{sh} y\right)(\tau)d\tau \le 0.$$

Sampled-data LTI systems case (*Fujioka 2009*)

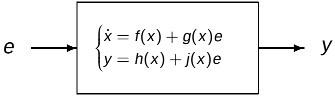
- *Properties of Δ_{sh} are used for the case of LTI sampled-data systems using IQC.*
- *Robust stability analysis via frequency based conditions.*
- *Conditions are constructive (LMIs) thanks to KYP lemma.*
- *Among the less conservative stability criteria.*

Can not be applied to bilinear systems

Objective: generalization to the bilinear case using dissipativity approach.

Dissipativity

- *An abstract extension of the notion of energy.*
- *Generalization of Lyapunov functions technique, for input-output systems.*
- *Encompass several properties of dynamical systems such as passivity and L_2-gain.*



$$\dot{V}(x(t)) \leq \mathcal{S}(y(t), e(t))$$

$V(x)$ *is a storage function* $\mathcal{S}(y, e)$ *is a supply rate.*

Properties of the operator

Boundedness property

For all $y \in L_2[t_k, t_{k+1})$ *and* $0 < X^* = X \in \mathbb{R}^{n\times n}$ *:*

$$\int_{t_k}^{t} (\Delta_{sh} y)^* X (\Delta_{sh} y)\, d\tau - \delta_0^2 \int_{t_k}^{t} y^* X y\, d\tau \le 0, \quad \forall t \in [t_k, t_{k+1})$$

Passivity property

For all $y \in L_2[t_k, t_{k+1})$ *and* $0 \le Y^* = Y \in \mathbb{R}^{n\times n}$*:*

$$\int_{t_k}^{t} (\Delta_{sh} y)^* Y y\, d\tau + \int_{t_k}^{t} y^* Y (\Delta_{sh} y)\, d\tau \le 0, \quad \forall t \in [t_k, t_{k+1})$$

$$\Rightarrow \int_{t_k}^{t} \underbrace{\begin{bmatrix} y \\ e \end{bmatrix}^T \begin{bmatrix} -\delta_0^2 X & Y \\ Y & X \end{bmatrix} \begin{bmatrix} y \\ e \end{bmatrix}}_{\text{supply rate } \mathcal{S}(y,e)} d\tau \le 0, \quad \forall t \in [t_k, t_{k+1})$$

Stability result

Local Analysis:

$$\mathcal{P} = conv\{x_1, x_2, \ldots, x_p\}$$

$$A[x(t_k)] \in \mathcal{P}_A = conv\{A_1, A_2, \ldots, A_p\}$$

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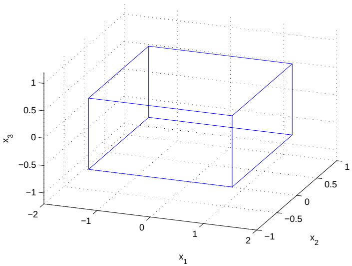


Stability result

Theorem (Automatica'14)

If there exist symmetric positive definite matrices X, Y, $P \in \mathbb{R}^{n\times n}$ and P_2, $P_3 \in \mathbb{R}^{n\times n}$, such that the following LMIs are satisfied

$$\begin{bmatrix} A_i^T P_2 + P_2^T A_i & P - P_2^T + A_i^T P_3 & P_2^T B \\ * & -P_3 - P_3^T + (\frac{2}{\pi} h_{max})^2 X & P_3^T B - Y \\ * & * & -X \end{bmatrix} < 0$$

$$\forall i \in \{1, 2, ..., p\},$$

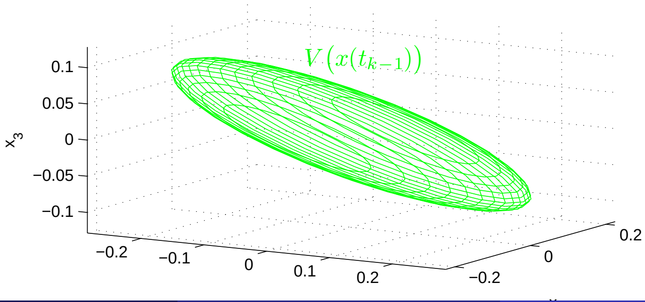
then the sampled-data system is locally asymptotically stable.

Stability result

Main idea:

dissipativity properties $\Rightarrow$ *contraction of sub-level sets defined by* $V(x) = x^T P x$:

$$\dot{V} < \mathcal{S}(y, e) \quad \Rightarrow$$



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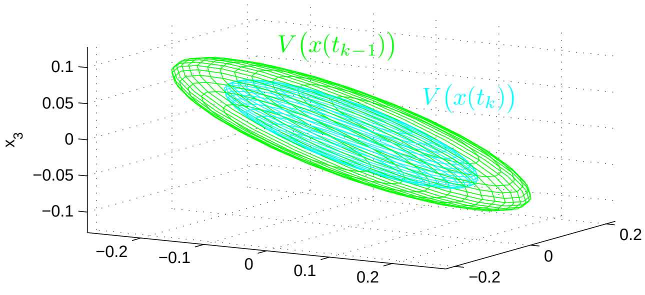

Stability result

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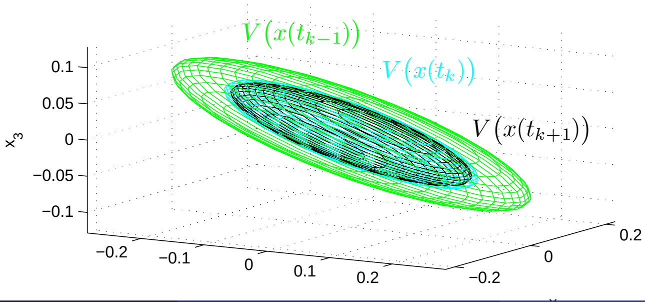
dissipativity properties $\Rightarrow$ *contraction of sub-level sets defined by* $V(x) = x^T P x$:

$$\dot{V} < \mathcal{S}(y, e) \quad \Rightarrow$$



Numerical example

Consider the example from (Amato et al 2007) and (Tarbouriech et al 2009):

$$A_0 = \begin{bmatrix} -0.5 & 1.5 & 4 \\ 4.3 & 6.0 & 5.0 \\ 3.2 & 6.8 & 7.2 \end{bmatrix}; \quad B_0 = \begin{bmatrix} -0.7 & -1.3 \\ 0 & -4.3 \\ 0.8 & -1.5 \end{bmatrix}$$

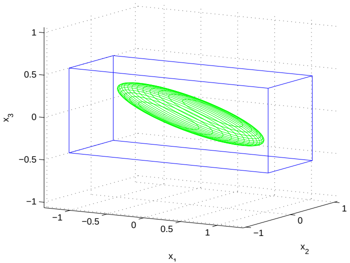
$$N_1 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}; \quad N_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$F = \begin{bmatrix} 0.0016 & 0.0035 & 0.0034 \\ 2.2404 & 3.2676 & 5.9199 \end{bmatrix}$$

$$\mathcal{P} = [-1.35, +1.35] \times [-0.5, +0.5] \times [-0.5, +0.5]$$

Numerical example

Region of attraction:

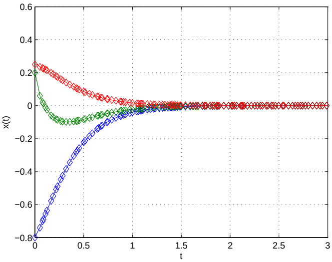


	Method 1 (CDC'12)	*Method 2 (CDC'12)*	*Theorem (ADHS'12)*	*Theorem (Automatica'14)*
h_{max}	5×10^{-3}	13×10^{-3}	43×10^{-3}	51×10^{-3}

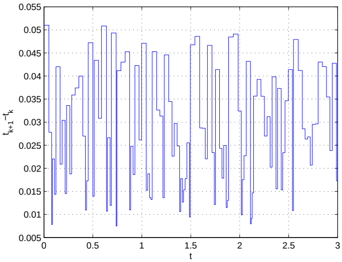
Numerical example

Simulation:

State evolution for the sampled-data bilinear system with $h_{max} = 51 \times 10^{-3}$*:*



State evolution

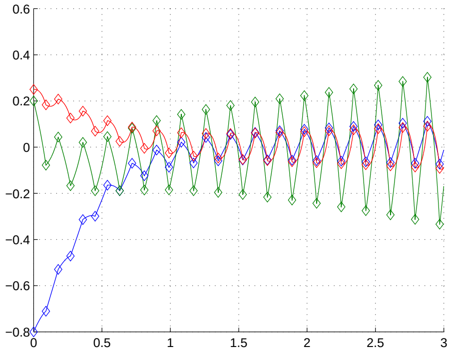


Time variations of the sampling intervals

Numerical example

Simulation:

State evolution for the sampled-data bilinear system with $t_{k+1} - t_k = 90 \times 10^{-3}$*:*



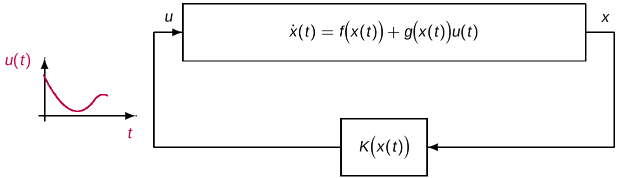
Summary

- *Stability of bilinear sampled-data systems was addressed using a robust control theory approach.*
- *The method is based on the analysis of contractive invariant sets, and it is inspired by the dissipativity theory.*
- *The proposed stability conditions are constructive via LMIs.*
- *How to generalize to a more general class of nonlinear systems?*

Outline

Problem formulation

Continuous-time controller

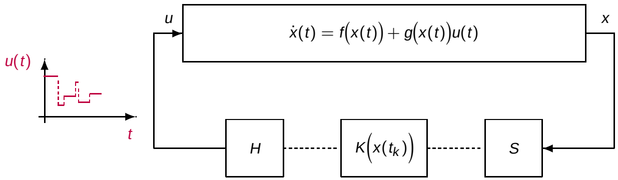


$$u(t) = K\Big(x(t)\Big)$$

The (continuous-time) controller K (of class $\mathcal{C}^1$) stabilizes asymptotically the origin of the system.

Problem formulation

Digital implementation under asynchronous sampling (emulation approach)



$$u(t) = K\big(x(t_k)\big), \quad \forall t \in [t_k, t_{k+1}), \qquad 0 < \epsilon \leq t_{k+1} - t_k \leq \underbrace{h_{max}}_{MASP}, \quad \forall k \in \mathbb{N}.$$

Problem: *Extend the dissipativity-based stability criteria for this more general class of nonlinear systems.*

Dissipativity-based representation

Equivalent model

Closed-loop system

$$\dot{x}(t) = f\big(x(t)\big) + g\big(x(t)\big)K\big(x(t_k)\big)$$

$$\dot{x}(t) = \underbrace{f\big(x(t)\big) + g\big(x(t)\big)K\big(x(t)\big)}_{f_n(x(t))} + \underbrace{g\big(x(t)\big)}_{g_n(x(t))}\underbrace{\Big(K\big(x(t_k)\big) - K\big(x(t)\big)\Big)}_{e(t)}$$

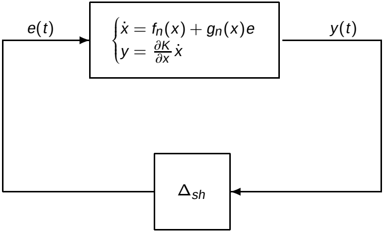
The system can be represented by the interconnection of:

$$\begin{cases} \dot{x}(t) = f_n\big(x(t)\big) + g_n\big(x(t)\big)e(t) \\ y(t) = \frac{\partial K}{\partial x}\dot{x}(t) \end{cases}$$

with the operator $\Delta_{sh} : y \to e$ defined by:

$$e(t) = (\Delta_{sh}\, y)(t) = -\int_{t_k}^{t} y(\tau)d\tau$$

Dissipativity-based representation



$$\begin{cases} \dot{x}(t) = & f_n(x(t)) + g_n(x(t))e(t) \\ y(t) = & \dot{x}(t) \end{cases}$$

$$(\Delta_{sh} y)(t) := -\int_{t_k}^{t} y(\tau) d\tau.$$

Stability analysis

Theorem (ECC'13)

Consider a neighbourhood $\mathcal{D} \subset \mathbb{R}^n$ of the origin $x = 0$ and differentiable positive definite function $V : \mathcal{D} \to \mathbb{R}^+$ psuch that there exist class $\mathcal{K}$ function β_1 et β_2 with:

$$\beta_1(|x|) \leq V(x) \leq \beta_2(|x|), \quad \forall x \in \mathcal{D}.$$

If for $\alpha > 0$ and for any $x(t) \in \mathcal{D}$, the function V satisfies:

$$\begin{aligned} \dot{V}\big(x(t)\big) + \alpha V\big(x(t)\big) &\leq \boldsymbol{S}\big(y(t), e(t)\big) \\ \dot{V}\big(x(t)\big) + \alpha V\big(x(t)\big) &\leq \boldsymbol{S}\big(y(t), e(t)\big) exp(-\alpha h_{max}) \end{aligned}$$

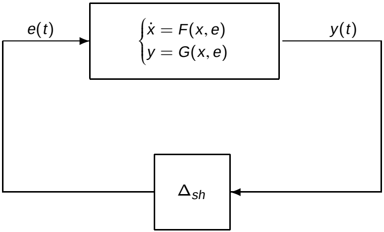
Then the equilibrium $x = 0$ of the sampled-data system is locally uniformly asymptotically stable. The maximal sub-level set of V that is contained in $\mathcal{D}$:

$$c^* = \max_{\mathcal{L}_c \subset \mathcal{D}} c, \qquad \mathcal{L}_c := \{x \in \mathbb{R}^n : V(x) \leq c\}.$$

is an estimate of the domain of attraction.

Polynomial systems

Specialization to the case of polynomial systems



$$F(x,e) := \underbrace{f(x) + g(x)K(x)}_{f_n(x)} + \underbrace{g(x)}_{g_n(x)}\, e$$

$$G(x,e) := \frac{\partial K}{\partial x} F(x,e)$$

$f(x)$, $g(x)$ and $K(x)$ polynomials $\Rightarrow$ $F(x,e)$ and $G(x,e)$ are polynomials

Polynomial systems

Specialization for the case of polynomial systems

$$\dot{V}\big(x(t)\big) + \alpha V\big(x(t)\big) \leq \boldsymbol{S}\big(y(t), e(t)\big)$$
$$\dot{V}\big(x(t)\big) + \alpha V\big(x(t)\big) \leq \boldsymbol{S}\big(y(t), e(t)\big) exp(-\alpha h_{max})$$

$$0 \leq -\frac{\partial V}{\partial x} F(x,e) - \alpha V(x) + \Big[-\delta_0^2 G^T(x,e) X G(x,e) + 2G^T(x,e) Y e + e^T X e \Big]$$
$$0 \leq -\frac{\partial V}{\partial x} F(x,e) - \alpha V(x) + \Big[-\delta_0^2 G^T(x,e) X G(x,e) + 2G^T(x,e) Y e + e^T X e \Big] exp(-\alpha h_{max})$$

The last inequalities are of the form $p(\xi) \geq 0$*, where* $p(\xi) \in \mathbb{R}[\xi]$*, and* $\xi = (x, e)$*.*

Verification of $p(\xi) \geq 0$ *is a difficult problem!* $\rightarrow$ *simplification using SOS*

A multivariate polynomial $p(\xi) \in \mathbb{R}[\xi]$ *is said to be a sum of squares (SOS) (Papachristodoulou 2005) if there exist* $p_i(\xi) \in \mathbb{R}[\xi]$*,* $i \in \{1, \ldots, M\}$*, such that* $p(\xi) = \sum_{i=1}^{M} p_i^2(\xi)$*.*

Polynomial systems

Corollary (ECC'13)

In the case where $f(x)$, $g(x)$ and $K(x)$ are polynomials, let

$$\mathcal{D} = \{x \in \mathbb{R}^n : \mu_l(x) \geq 0,\ l = 1, 2, \dots, s\}$$

be a neighbourhood of $x = 0$. Suppose that there exist a polynomial function $V(x) \in \mathbb{R}[x]$ and sums of squares $\sigma_l(\xi)$ and $\varsigma_l(\xi)$, with $l \in \{1, \dots, s\}$ and $\xi = (x, w)$, such that the following polynomials are SOS:

- $V(x) - \varphi(x)$,
- $-\sum_{l=1}^{s} \sigma_l(\xi)\mu_l(x) - \frac{\partial V}{\partial x}F(x,e) - \alpha V(x) + \left[-\delta_0^2 G^T(x,e)XG(x,e) + 2G^T(x,e)Ye + e^T Xe \right]$,
- $-\sum_{l=1}^{s} \varsigma_l(\xi)\mu_l(x) - \frac{\partial V}{\partial x}F(x,e) - \alpha V(x) + \left[-\delta_0^2 G^T(x,e)XG(x,e) + 2G^T(x,e)Ye + e^T Xe \right] e^{-\alpha h_{max}}$.

Then $x = 0$ is locally uniformly asymptotically stable. Moreover, $\mathcal{L}_{c^*}$ is an estimate of the domain of attraction.

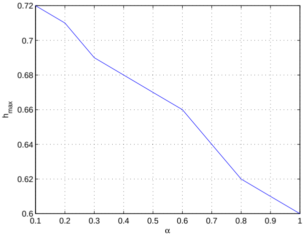
Numerical example

Consider the example from (Nešić et al 2009)

$$\dot{x} = dx^2 - x^3 + u, \quad u = K(x) = -2x, \quad \text{with } |d| \leq 1.$$

	ECC'13	(Nešić et al 2009)	(Karafyllis et al 2007)
h_{max}	0.72	0.368	0.1428

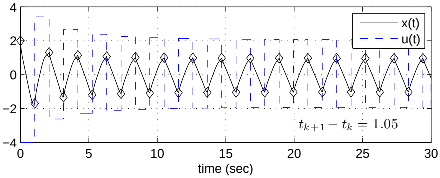
Trade-off between the decay rate α and the MASP:

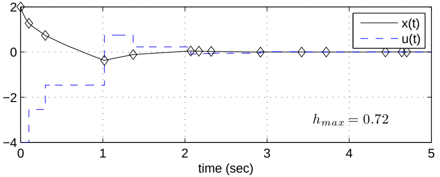


Numerical example

Simulation:

State evolution for the sampled-data system:





Summary

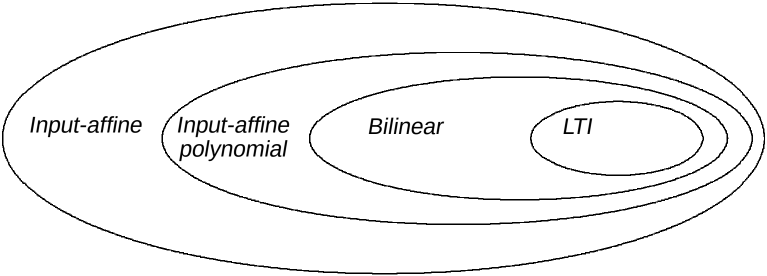
- *Address a quite general class of systems thanks to exponential dissipativity.*
- *Sufficient conditions for the stability of nonlinear sampled-data systems, which are affine in the control.*
- *The results are numerically tractable for the case of polynomial systems, with the use of SOS.*

Outline

Conclusions

- *A contributions to the stability analysis of nonlinear systems under aperiodic sampling.*
- *A particular attention has been given to the case of bilinear systems:*
    - *Hybrid system modeling approach*
    - *Dissipativity approach*
- *Extend the dissipativity-based results developed for bilinear systems to a more general class of nonlinear systems.*
- *The methods provide quantitative estimates of the MASP.*

Conclusions



Perspectives

- *Include other network-imposed imperfections such as: time-varying delays, protocols .. etc.*
- *Improve the numerical solvability of the proposed conditions in order to decrease the conservatism.*
- *Controlled sampling: event-based control, self-triggered control and state-dependent sampling (Fiter et al 2012), for nonlinear systems.*

Personal publications

Journals:

- *H. Omran, L. Hetel, J.-P. Richard, and F. Lamnabhi-Lagarrigue. "**Stability analysis of bilinear systems under aperiodic sampled-data control**". Automatica. 2014.*
- *H. Omran, L. Hetel, J. P. Richard, and F. Lamnabhi-Lagarrigue. "**Stabilité des systèmes non linéaires sous échantillonnage apériodique**". Journal Européen des Systèmes Automatisés. Accepted. Selected work from 5èmes Journées Doctorales / Journées Nationales MACS. 2013.*

Conferences:

- *H. Omran, L. Hetel, and J.-P. Richard. "**Local stability of bilinear systems with asynchronous sampling**". In The 4th IFAC Conference on Analysis and Design of Hybrid Systems(ADHS), pages 19-24, 2012.*
- *H. Omran, L. Hetel, J.-P. Richard, and F. Lamnabhi-Lagarrigue. "**Stability of bilinear sampled-data systems with an emulation of static state feedback**". In IEEE 51st Annual Conference on Decision and Control (CDC), pages 7541-7546, 2012.*
- *H. Omran, L. Hetel, J.-P. Richard, and F. Lamnabhi-Lagarrigue. "**On the stability of input-affine nonlinear systems with sampled-data control**". In European Control Conference (ECC), pages 2585-2590, 2013.*

Thank you for your attention